Some old ideas challenged

Until the time of Galileo, the motion of a projectile was based on the teachings of Greek philosopher Aristotle. For example, Albert of Saxony (1316–90), rector of Paris University, taught that the trajectory of a projectile was in three parts: firstly, the upward motion where the initial impetus suppressed gravity; secondly, a period where the projectile's impetus and gravity were compounded; and thirdly, when gravity and air resistance overcame the natural impetus. This produced a trajectory as shown in Figure 5.7.

It wasn't until 1638 that the trajectory of a projectile could be described mathematically. Galileo's description proved to be correct and has been the basis of mechanics since. The mathematical techniques that Galileo pioneered, later refined by Newton, can be seen in the examples that follow.

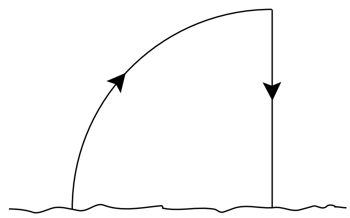

Figure 5.7
Until the 1600s, people thought that projectile motion was more like this.

Example

The L16 mortar is a weapon currently used by Commonwealth defence forces. If a mortar shell was fired at 200 m s^{-1} at an angle of 40° to the ground, calculate:

(a) the initial vertical and horizontal components of the velocity;
(b) the maximum height reached;
(c) the time of flight (total time taken from start to finish);
(d) the horizontal range;
(e) the impact velocity.

Solution

Let the upward direction be positive: $\boldsymbol{a} = -10$ m s^{-2}.

(a)
- Vertical: $\boldsymbol{u}_v = \boldsymbol{v} \sin\theta = 200 \times \sin 40° = +129$ m s^{-1} in positive direction (up).
- Horizontal: $\boldsymbol{u}_h = \boldsymbol{v} \cos\theta = 200 \times \cos 40° = 153$ m s^{-1}.

(b) At maximum height $\boldsymbol{v}_v = 0$ m s^{-1}.

$$(\boldsymbol{v}_v)^2 = (\boldsymbol{u}_v)^2 + 2\boldsymbol{a}\boldsymbol{s}_v, \text{ hence } \boldsymbol{s}_v = \frac{\boldsymbol{v}^2 - \boldsymbol{u}^2}{2\boldsymbol{a}} = \frac{0^2 - (+129)^2}{2 \times -10} = +832 \text{ m}$$

(c) Time of flight can either be calculated by **(i)** determining the time taken to reach maximum height ($\boldsymbol{v} = 0$) and doubling it; or **(ii)** determining time taken until final vertical velocity is equal and opposite to initial vertical velocity; or **(iii)** until vertical displacement is zero again.

By **(i)** $\boldsymbol{v}_v = \boldsymbol{u}_v + \boldsymbol{a}t$, hence $t = \dfrac{\boldsymbol{v} - \boldsymbol{u}}{\boldsymbol{a}} = \dfrac{0 - (+129)}{-10} = 12.9$ seconds. Total time = 25.8 s.

By **(ii)** $\boldsymbol{v}_v = \boldsymbol{u}_v + \boldsymbol{a}t$, hence $t = \dfrac{\boldsymbol{v} - \boldsymbol{u}}{\boldsymbol{a}} = \dfrac{-129 - (+129)}{-10} = 25.8$ s.

By **(iii)** $\boldsymbol{s}_v = \boldsymbol{u}_v t + \frac{1}{2}\boldsymbol{a}t^2$, hence $0 = +129t + -5t^2$; $5t = 129$; hence $t = 25.8$ s.

(d) Horizontal range = horizontal component of initial velocity × time of flight.

$$\boldsymbol{s}_h = \boldsymbol{v}_h \times t = 153 \times 25.8 = 3947 \text{ m}$$

(e) The impact velocity will have the same magnitude as the initial velocity, but will be directed generally downward not up. The angle of impact (θ) will be the same as the angle of elevation (40°). Thus, the impact velocity is 200 m s^{-1} at an angle 40° to the horizontal.

NOVEL CHALLENGE

Acapulco cliff divers jump off a cliff 35 m high and just miss rocks 5 mm out from the base.
What is their minimum push-off speed?

NOVEL CHALLENGE

On the Moon, astronauts hit a golf ball 180 m. If they hit the same ball on Earth with the same speed and angle, how far will it go (neglect air resistance)? Note $\boldsymbol{g}_{moon} = 1.6$ m s^{-2}. By the way, there are three golf balls still on the Moon. Learn this off by heart — it could be useful.

Complementary angles of elevation

The range of a projectile fired at an elevation angle of 40° will also be the same if it is fired at 50°. The angles 40° and 50° are called **complementary angles** because they add up to 90°. Other examples of complementary pairs are: 30° and 60°; 20° and 70° etc. In other words, the range of a projectile will be the same for elevation angles of θ and 90° – θ. It is interesting that sin θ = cos (90° – θ).

Activity 5.2 TOY CANNON

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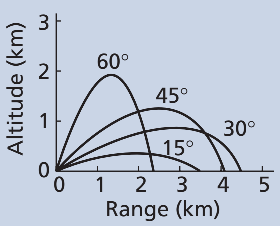
If you have access to a toy cannon, try firing some projectiles at complementary angles and collect some data. Perhaps you could design a device that uses a rubber band, a mousetrap or a spring to fire small objects up an incline. Then you could vary the elevation angle. Whatever you do, you should aim to confirm or refute the above assertion about complementary angles.

NOVEL CHALLENGE

The following graphs show how the range and altitude of a projectile changes with elevation angle **in the presence of air**. Plot a graph of maximum altitude versus elevation angle and predict maximum altitude for an angle of 90°. *Should the graph pass through the origin (0,0)? Why?*



Example

In the earlier example, an elevation angle of 40° produced a range of 3947 m. If the theory is correct, then an angle of 50° should produce the same range.

(a) Prove this assertion.
(b) By how much do the times of flight differ?
(c) Do the impact velocities differ? (The initial velocity was 200 m s^{-1}.)

Solution

(a) Let $\boldsymbol{a}$ = –10 m s^{-2}.

$$\boldsymbol{u}_v = \boldsymbol{v} \sin\theta = 200 \times \sin 50° = +153 \text{ m s}^{-1} \text{ (upward)}$$
$$\boldsymbol{u}_h = \boldsymbol{v} \cos\theta = 200 \times \cos 50° = 129 \text{ m s}^{-1}$$

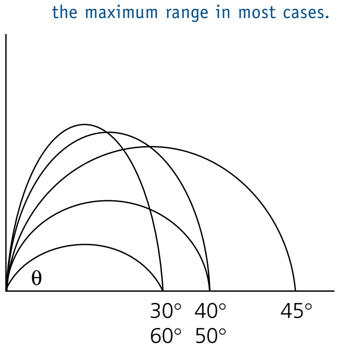
Impact velocity in vertical direction ($\boldsymbol{v}_v$) = $-\boldsymbol{u}_v$ = –153 m s^{-1}.
Hence, the range is identical.

$$\boldsymbol{v}_v = \boldsymbol{u}_v + \boldsymbol{a}t, \text{ hence } t = (\boldsymbol{v}_v - \boldsymbol{u}_v/\boldsymbol{a}) = (-153 - +153)/10 = 30.6 \text{ s}$$
$$\boldsymbol{s}_h = \boldsymbol{v}_h \times t = 129 \times 30.6 = 3947 \text{ m}$$

(b) The times of flight were: for 40°, t = 25.8 s; for 50°, t = 30.6 s; difference was 4.8 s.
(c) Impact velocities are different but only in direction not magnitude.
For 40°, $\boldsymbol{v}_{impact}$ = 200 m s^{-1} at 40° to horizontal.
For 50°, $\boldsymbol{v}_{impact}$ = 200 m s^{-1} at 50° to horizontal.

Maximum range

Figure 5.8
An elevation angle of 45° produces the maximum range in most cases.



It was the invention of the cannon in the late 1400s that created a new form of warfare. War at sea using cannons became more common and defence using medieval castles became obsolete. Medieval mechanics also became obsolete. Until then, the motion of a projectile was only of philosophical interest because they all thought they knew how projectiles moved — after all, Aristotle described the motion over 1000 years earlier and no one was prepared to challenge his theories. The theories weren't challenged until they had to be tested in warfare and were found wanting. Aiming was very much a hit-or-miss affair; there was no way of determining the trajectory or even the angle of launch in advance. It wasn't until self-taught engineer Niccolo Fontana published the results of his experiments in 1546 that gunners realised a 45° angle of elevation would give the maximum range.

In Figure 5.8 the maximum range can be calculated by letting θ = 45°. In this case the horizontal and vertical components of the initial velocity both equal 141 m s^{-1}, the time of flight equals 28.2 s and the maximum range works out to be 3976 m.