Examination session: May 2023

IB extended essay in physics

How does viscosity affect the flowrate of a working fluid and what are the limitations of Hagen-Poiseuille law?

Word Count: 3922

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1. Introduction:

1.1 Motivation:

The power output of a hydroelectric generator depends on many parameters: the friction factor, turbine head loss, generator winding, etc, However, there is one factor that people tend to ignore: viscosity of the fluid, namely, the resistance of a fluid to change in motion. Consequently, the viscosity affects the rate at which the fluid flowrates through the generator, affecting its power output. Therefore, in this extended essay, I hope to investigate the physics behind the flow rate.

1.2 The Hagen-Poiseuille law:

For this paper, to investigate the effect of viscosity on the flowrate, we will be looking at a specific law in physics: the Hagen-Poiseuille law. This law could be derived from the Navier–Stokes momentum equations in 3D cylindrical coordinates (r, θ, x). Under this, the equation is derived to be:

$$Q = \frac{\Delta P \pi R^4}{8 \mu L} \tag{1}$$
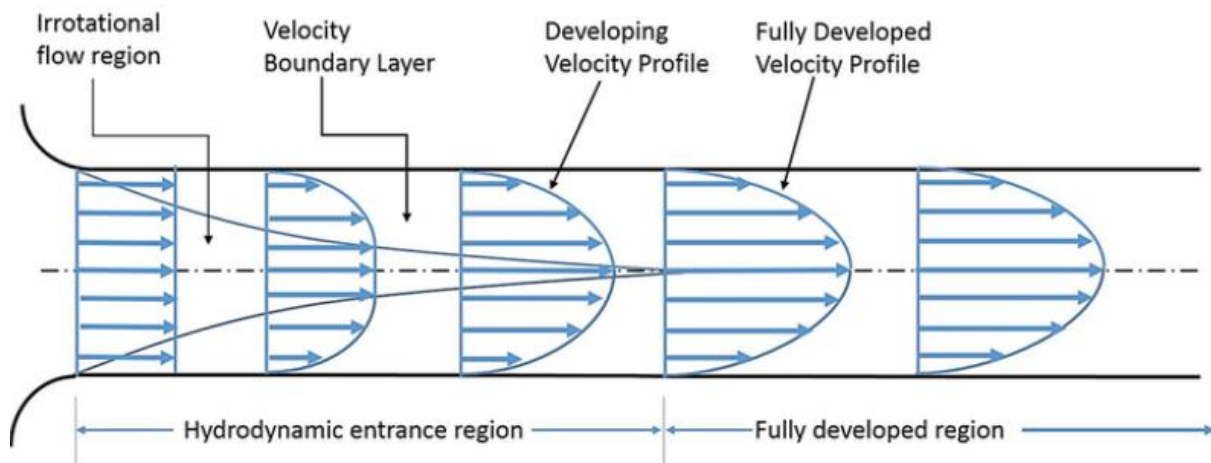
However, there are a few constraints:

Figure 1: Hagen-Poiseuille flow (Tec-Science 2021)

- This law only works when the length of the pipe is relatively long compared with its diameter. The reason is that this law assumes parabolic flow. As shown in the image above, when the pipe is too

short, it will still be in the phase of inlet flow when the fluid comes out of the other side, which will invalidate the law. In other words, the flow must be steady and fully developed (Tec-Science, 2019).

- The flow must be laminar: that is, the Reynold's number (R_e) is smaller than 2300. When the Reynold's number is bigger than 2300, the flow will then be transition or turbulent, meaning that the pressure drop across the pipe will be higher than expected (*SimScale 2021*).

- The fluid must be Newtonian, meaning that the viscosity of the fluid remains constant no matter how much shear is applied. Liquids such as oobleck, ketchup are non-Newtonian and therefore invalid for the law.

- This law is usually worked for fluid flowing in a pipe from a reservoir: this implies that the water level above is constant, as is the pressure. However, in our experimental set-up, the height is time-dependent, so will be the pressure. We will see how the Hagen-Poiseuille law will work with a changing pressure.

1.3 Variables:

There are a few experimental preparations and observations that are essential to the procedure.

The dependent variable tested is the average flowrate of water and 5 different concentrations of sucrose solutions. They are being tested how fast 470ml of each goes out of a pipe.

The independent variable is the viscosity of the fluid. I chose water and 5 different concentration of sucrose solution, ranging from 30% to 65% concentration. Thus, 6 totally different viscosities were tested.

Control variables

Variables	How it was controlled
Viscosity of the fluid	The viscosity of the fluid needs to be kept under the same temperature as it has a dramatic effect on viscosity. The same granulated Rogers sugar to make solutions was used.
Temperature	Unfortunately, not all trials could have been kept at the same temperature. Therefore, I used the Arrhenius equation to adjust the values of viscosity as temperature changes and kept the effect of temperature constant in a sense.
Length, radius of the pipe	These dimensional properties of the pipe were all easily kept constant as the same pipe was used throughout.
Motion sensor	The same motion sensor was used throughout the experiment.

2. Experimental procedure

2.1 Experimental set-up:

Figure 2, the whole experimental set-up
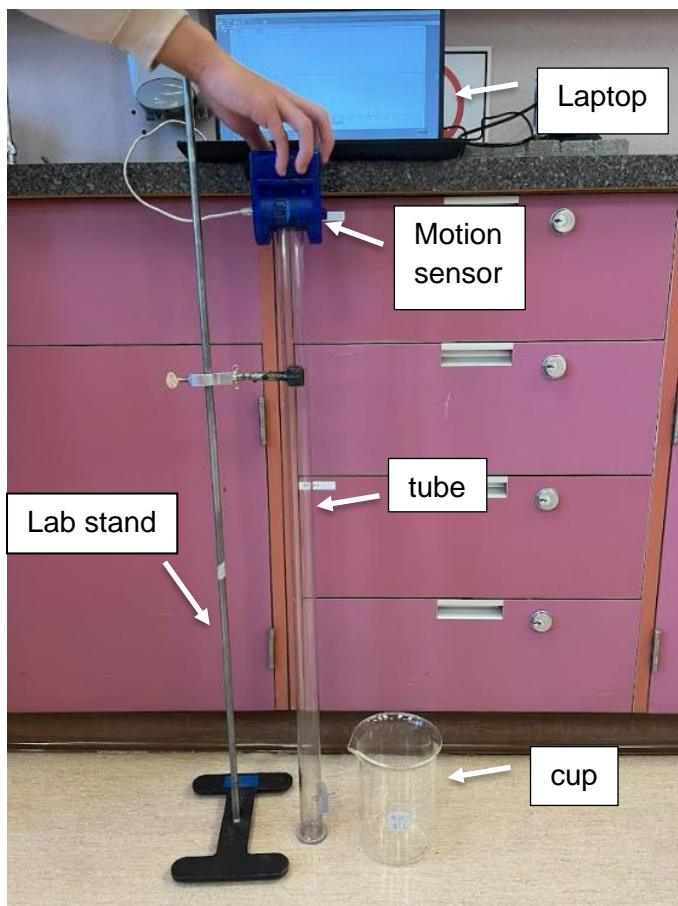
The tube is fixed by a lab stand. Using my fingers to block the bottom of the pipe, I pour in the liquid until the black mark, which is measured to be 470ml. The, putting the collection rate at 20Hz, I start the PASCO motion sensor with position vs time graph collection and place the sensor on top of the tube. Now I can unblock the hole and a position vs time graph will start recording. I have put a bowl in front of the pipe to collect the fluid so I could reuse for the next trial.

During the collection period, it is important for me to wait until the flow fully stops. Otherwise, because the flowrate decreases over time, it will be

easy to think the flow stopped although it has not. For high viscosity fluid, the flow will be very slow, and it was important to be patient.

For each different viscosity, the trials were repeated about 8 times on average. This is because each trial is relatively time-consuming, and errors do rise during the trials. The common errors include but not limited to: the motion sensor not registering properly, wobbling tube and wobbling cup. Therefore, I will always have to eliminate 1-2 defective trials, and take 6 valid trials in the end. The same procedure was repeated with 6 different liquids of different viscosities.

Figure 3: a closer look at the motion sensor on top of the tube

This particular methodology using a motion sensor is chosen because a motion sensor can be easily obtained in a physics classroom. Video analysis was also attempted but the motion gave a better result. The PASCO motion sensor allowed me to perform slope and line of best-fit calculations on the data. For video analysis, it was also very time-consuming to transfer all the videos from the phone onto the laptop.

Figure 4: Experiment in process:
water flowing out of the tube

Figure 5: Measuring 300g of sugar to
make 30% sucrose solution

2.2 Time taken for emptying:

To determine the time taken, we look at the position versus time graph of the fluid level. The time between the two flat ends is the time taken, when the fluid stops flowing. Below is an example graph taken from one of the 50% sucrose solution trial.

Figure 6: An example taken from the 50% sucrose solution trial
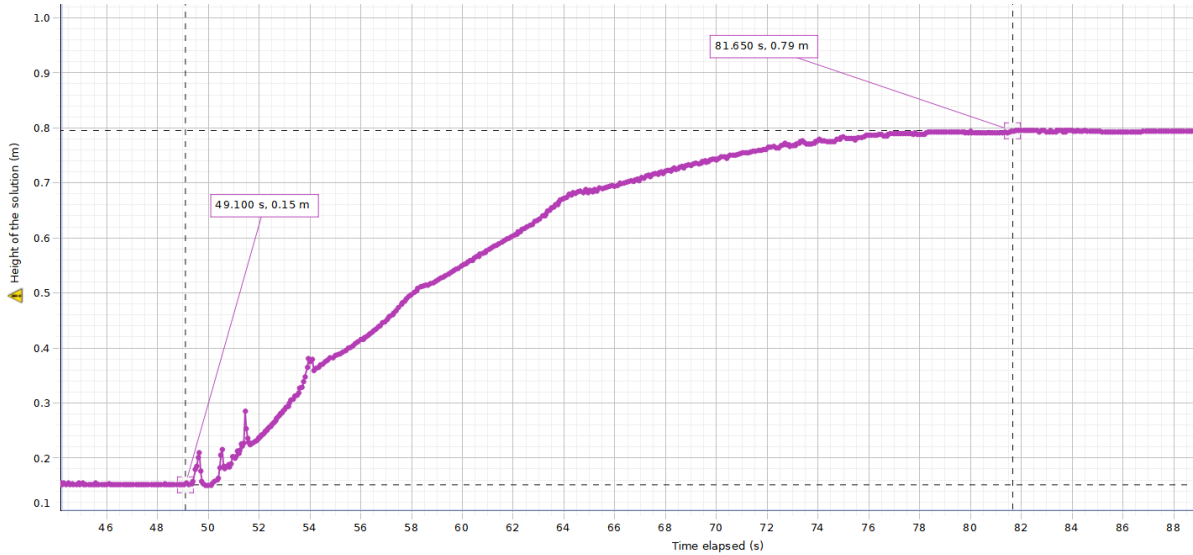
It might initially seem difficult to read where the endpoints really are from this graph, but as we zoom in separately, the endpoints are very clear.

The starting point of the trial: 49.10s

Figure 7: The starting point of the trial (zoomed in)
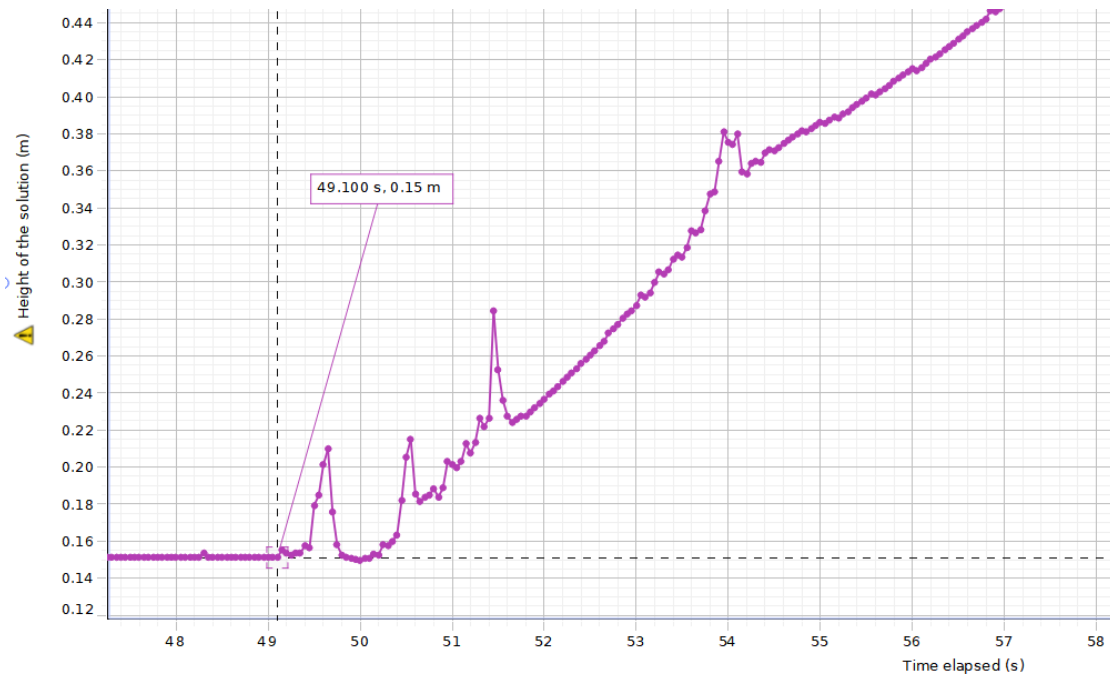
The PASCO motion sensor tends to have a small wobble at the start of each trial. However, from reading this graph, it is still obvious when the trial started: the initial flat line at 49.10s.

The endpoint of the trial: 81.65s

Figure 8: The ending point of the trial (zoomed in)
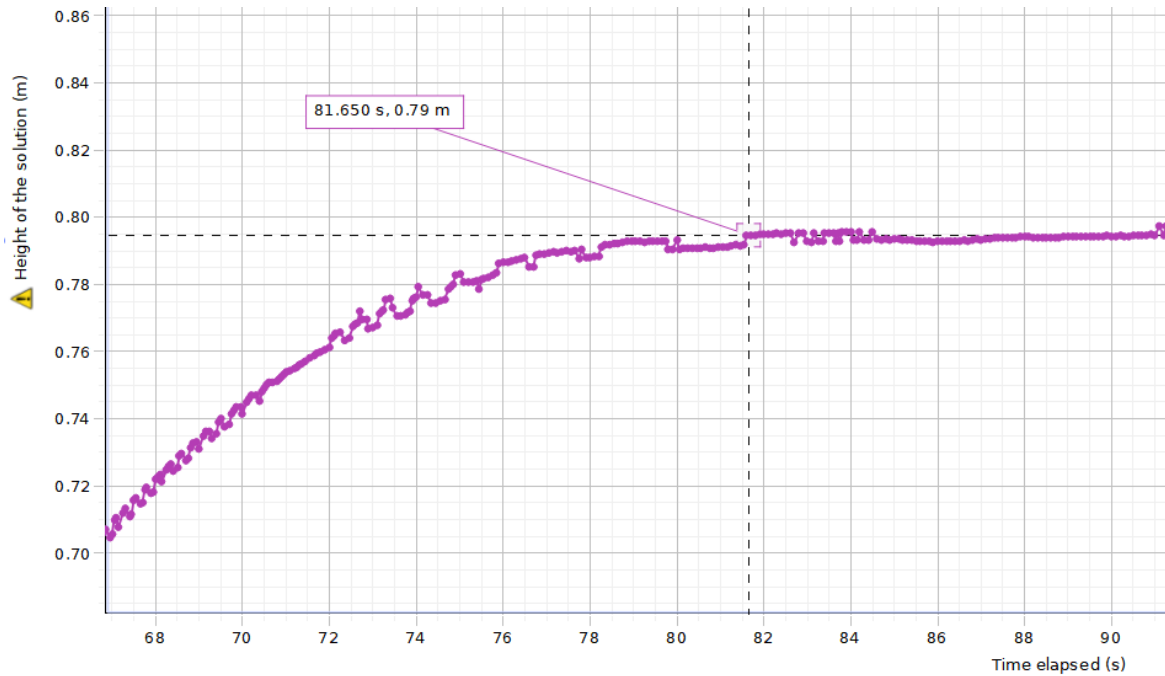
The endpoint is fairly obvious as it is zoomed in. The height of the solution clearly stopped changing after t = 81.65 seconds. The final time is then subtracted by the initial starting time to find the total time taken.

$$\Delta t = \text{Time taken} = 81.65 - 49.10 = 32.55s$$

3. Experimental data

3.1 Raw Data:

Table 1: Experimental data of each trial

Type of fluid	Time taken for trial 1, T_1 (s) (±0.05s)	Time taken for trial 2, T_2 (s) (±0.05s)	Time taken for trial 3, T_3 (s) (±0.05s)	Time taken for trial 4, T_4 (s) (±0.05s)	Time taken for trial 5, T_5 (s) (±0.05s)	Time taken for trial 6, T_6 (s) (±0.05s)
water	18.10	18.15	18.10	18.45	18.20	18.35
30% sucrose solution	22.75	22.10	23.00	22.25	22.55	22.65
50% sucrose solution	32.20	31.90	32.15	32.55	31.75	32.15
55% sucrose solution	39.80	40.10	39.75	40.35	39.35	39.55
60% sucrose solution	49.75	48.75	49.10	48.00	48.90	50.00
65% sucrose solution	113.45	108.95	109.55	116.15	111.95	110.30

The uncertainty in the time is chosen to be 0.05s because the sample rate of the motion sensor was 20.00Hz.

1 second ÷ 20, this means that each datapoint is 0.05s away from each other and it is the lowest precision of time.

We will process the data and put them in *Table 2*. We will find the average of time taken, and the 50% sucrose solution will be used as an example to calculate for the average value of all the trials for a single solution,

$$T_{average} = \frac{Time\ taken\ for\ trial\ 1 + Time\ taken\ for\ trial\ 2 + \cdots}{6}$$

$$T_{average} = \frac{32.30 + 31.90 + 32.15 + 32.35 + 31.75 + 32.15}{6} = 32.08s$$

To find the uncertainty, we calculate for the standard deviation,

$$\sigma = \sqrt{\frac{\sum_{n=1}^{N}(x_n - \mu)^2}{N}}$$

μ is the average of the trials, n is the trial number, x_n is s the value of trial n, and N is the size of the sample (or total number of trials). Hence, σ for the sample calculation can be found as follows:

$$\sigma = \sqrt{\frac{(32.2 - 32.08)^2 + (31.9 - 32.08)^2 + (32.15 - 32.08)^2 + (32.35 - 32.08)^2 + (31.75 - 32.08)^2 + (32.15 - 32.08)^2}{6}}$$

$$\sigma = 0.2$$

3.2 Viscosity:

The viscosity of the sucrose solution was found from online sources. The values come from the *International Journal of Food Properties* and the *ISCOTBALES*, an reliable online database for engineers. The values were then cross-checked and proven to be correct. The sucrose solutions on the papers were indicated to be weight/weight. For example, for a 55% solution, it will be 550g of sugar and 450g of water. In addition, the paper gave out the uncertainties of viscosities to be 5%.

Unfortunately, not all the trials were performed at 20℃ (293.15k) as the room temperature could not be always controlled. This happened for the 55%, 60%, 65% sucrose solution, in which it was at 22℃ (295.15k) instead of 20℃. Temperature can affect the viscosity and we will be adjusting the values accordingly.

The viscosity could be adjusted using the following equation, the Arrhenius model (*V.R.N.Telis, All, et al, 2006)*. E_a stands for the activation energy, R is gas constant, T is temperature of the sucrose solution, T_{ref} is the reference temperature of the sucrose solution (20℃) and μ is the viscosity.

$$\mu = \mu_{ref}\,exp\left[\frac{E_a}{R}\left(\frac{1}{T} - \frac{1}{T_{ref}}\right)\right] \tag{2}$$

Sample calculation using 60% sucrose solution,

$$\mu = 56.76\,exp\left[\frac{40696.29}{8.31}\left(\frac{1}{295.15} - \frac{1}{293.15}\right)\right]$$

$$\mu = 0.05069 Pa \cdot s$$

This is the corrected viscosity that we will be using for the calculation. The same correction will be done for other fluids as well.

3.3 Density of fluids:

According to the Hagen-Poiseuille equation, the density is a variable in it. It is also dependent on the temperature. However, no density adjustment will be made because first, there is not a formal equation to model the change. Second, the percentage difference is small, as the density of water at 21℃ and 26.7℃ *(USGS 2018)* is found to be 0.13%. A difference small enough to be ignored. The density at different sucrose solutions were calculated by me and cross-checked with external sources *(ISCOTABLES 7ed),* a reliable data booklet for scientists and engineers. Using the 55% sucrose solution as an example: The values are 550g of sugar with density $1.59 g/cm^3$ and 450g of water with density $1 g/cm^3$. Similarly, for the 60% solution, it will be 600g of sugar, 400g of water, and so on.

$$\rho_{solution} = \frac{M_{total}}{V_{total}} = \frac{M_{sugar} + M_{water}}{\frac{M_{sugar}}{\rho_{sugar}} + \frac{M_{sugar}}{\rho_{sugar}}} = \frac{550 + 450}{\frac{550}{1.59} + \frac{450}{1}} = \frac{1.258g}{cm^3} = 1258 kg/m^3$$

The density is $1258 kg/m^3$, and it matches with the external sources *(ISCOTABLES 7ed).*

3.4 Flowrate:

For each trial, the volume of the of fluid was kept at $(470\pm5) ml$.

The flowrate formula is given below:

$$Flowrate = \frac{\Delta volume}{\Delta time}$$

Thus, to determine the flowrate, we calculate using the formula below, and note that the volume is the same for all the trials. The time is the average time determined from the previous calculations:

$$Flowrate = \frac{\Delta volume}{T_{average}} = \frac{470 \pm 5}{32.08 \pm 0.2} = \frac{470}{32.08} \pm \frac{470}{32.08}\left(\frac{5}{470} + \frac{0.2}{32.08}\right) = (15.59 \pm 0.25)cm^3/s$$

$$\approx (15.6 + 0.3)cm^3/s \approx (0.0000156 \pm 0.0000003)m^3/s$$

It might seem weird to keep the values in m^3/s instead of cm^3/s. However, with the theoretical model that I am about to derive, it will be much easier to keep these values in standard units.

3.5 Processed data:

Table 2: All the processed data with some of their standard deviation (errors)

| Type of fluid	Density, ρ, (kg/m^3)	Viscosity, μ, (Pa· s) $\pm 5\%$	Average time taken, $T_{average}$, (s)	Standard deviation of time taken, σ, (s)	Flowrate, Q, (m^3/s)	Standard deviation in the flowrate, σ, (m^3/s)
Water	998	0.001	18.23	0.1	0.0000274	4×10^{-7}
30% sucrose solution	1128	0.003	22.55	0.3	0.0000222	5×10^{-7}
50% sucrose solution	1230	0.015	32.08	0.2	0.0000156	3×10^{-7}
55% sucrose solution	1255	0.025	39.82	0.3	0.0000126	2×10^{-7}
60% sucrose solution	1286	0.051	49.08	0.7	0.0000096	2×10^{-7}
65% sucrose solution	1316	0.130	111.73	2.5	0.0000042	2×10^{-7}

3.6 Additional notes:

Range of data: The reason that the data didn't go beyond 65% concentration is that it was physically impossible to mix 70% sucrose solution. In addition, if the fluid is too viscous, it will flow so slow that it will be difficult for the motion sensor to register the change in height. This problem already slightly occurred for the 65% concentration, and so the 70% will be even worse. The lowest concentration is chosen at 30% because any concentration lower than that have a viscosity that is too similar with water.

4. Theoretical models

4.1 Theoretical model:

Figure 9: Sketch of the set-up (Drakaki & Sianoudis 2008)
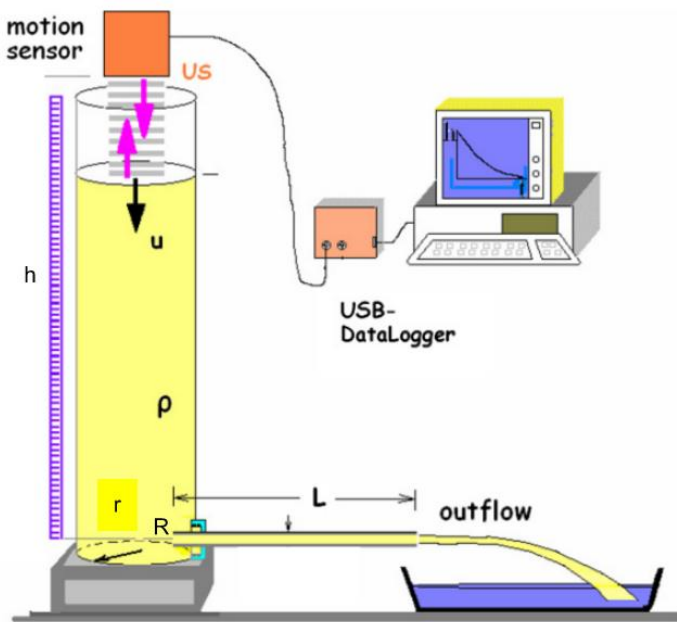
Here is a clarification of the terms in *Figure 7* and some that will be used in the upcoming derivation:

The radius of the tube (r)

Density of fluid (ρ)

Radius of the pipe (R)

Length of the pipe (L)

$Cross - sectional\ area\ of\ the\ tube\ (A)$

Initial height of the fluid in the tube (h_0)

viscosity of the fluid (μ)

Now starting off with the Hagen-Poiseuille law *(eq.1)*,

$$Q = \frac{\Delta P \pi R^4}{8\mu L}$$

And combine with the pressure given as a function of height and dependent on time,

$$\Delta P = \rho g h(t) \tag{3}$$

$$Q = \frac{\rho g h(t) \pi R^4}{8\mu L} = -\frac{dh}{dt}A, \text{ where } A = \frac{\pi r^2}{4} \tag{4}$$

Integrating both sides,

$$\int -\frac{\rho g \pi R^4}{8\mu LA} dt = \int \frac{dh}{h(t)} \tag{5}$$

$$\ln h(t) = -\frac{\rho g \pi R^4}{8\mu LA} t + C \tag{6}$$

At $t = 0s$, C is equal to h_0,

$$h(t) = h_0 e^{-\frac{\rho g \pi R^4}{8\mu LA} t} \tag{7}$$

Differentiating the equation,

$$\frac{dh}{dt} = h_0 (-\frac{\rho g \pi R^4}{8\mu LA}) e^{-\frac{\rho g \pi R^4}{8\mu LA} t} \tag{8}$$

To find the flowrate, we multiply it by the area (A),

$$Flowrate = Q(t) = A\frac{dh}{dt} = h_0 (\frac{\rho g \pi R^4}{8\mu L}) e^{-\frac{\rho g \pi R^4}{8\mu LA} t} \tag{9}$$

As we know, the flowrate that we got was the average flowrate, not the instantaneous one. To find the

average, we do the following. This is the equation for the <u>average flowrate.</u>

$$Flowrate_{average} = \frac{1}{t_{final} - t_{initial}} h_0 (\frac{\rho g \pi R^4}{8\mu L}) \int_{t=t_i}^{t=t_f} e^{-\frac{\rho g \pi R^4}{8\mu LA} t} \tag{10}$$

$$= \frac{1}{t_{final} - t_{initial}} h_0 (\frac{\rho g \pi R^4}{8\mu L}) (\frac{e^{-\frac{\rho g \pi R^4}{8\mu LA} t_{final}}}{-\frac{\rho g \pi R^4}{8\mu LA}} - \frac{e^{-\frac{\rho g \pi R^4}{8\mu LA} t_{initial}}}{-\frac{\rho g \pi R^4}{8\mu LA}}) \tag{11}$$

$t_{initial}$ is always 0s, so the latter part is just equal to 1.

$$Flowrate_{average} = \frac{1}{t_{final}} h_0 (-A)(e^{-\frac{\rho g \pi R^4}{8\mu LA} t_{final}} - 1) \tag{12}$$

In this equation, the density ρ, viscosity μ and time taken t vary from fluid to fluid. The rest of the

variables are all constant.

This result is very notable. For this derivation, a time-dependent Hagen-Poiseuille law is derived. The

relationship is exponential and affected by three variables. However, notice that density and viscosity are

somewhat related, and the density variable has minimal effect on the equation. The final relationship is $Q \propto$

$\frac{1}{t_{final}} (e^{-\frac{t_{final}}{\mu}} - 1)$ instead of $Q \propto \frac{\rho}{\mu}$ as according to the Hagen-Poiseuille law.

4.2 Theoretical data:

Using the average flowrate formula *(eq.12),* we verify using the measured variables. A sample calculation is done with the 50% solution. The following quantities, the dimensions of the tube, were measured using a Vernier calliper. The variables could be referred to *Figure 7.*

$$The\ radius\ of\ the\ tube, (r) = (0.030 \pm 0.005)m$$

$$Density\ of\ fluid, (\rho) = 1228kg/m^3$$

$$Radius\ of\ the\ pipe, (R) = (0.0020 \pm 0.0005)m$$

$$Length\ of\ the\ pipe, (L) = (0.060 \pm 0.005)m$$

$$Cross-sectional\ area\ of\ the\ tube, (A) = \frac{\pi r^2}{4} = \frac{\pi(0.030 \pm 0.005)^2}{4}$$

$$Initial\ height\ of\ the\ fluid\ in\ the\ tube, (h_0) = (0.62 \pm 0.01)m$$

$$Viscosity\ of\ the\ fluid, (\mu) = (0.0150 \pm 5\%)Pa \cdot S$$

$$Flowrate_{average} = \frac{1}{32.08}0.62(-\frac{\pi(0.03)^2}{4})(e^{-\frac{(1228)(9.81)\pi(0.002)^4}{8(0.015)(0.06)\frac{\pi(0.03)^2}{4}}(32.08)} - 1)$$

Using GDC, the final result is: $0.0000134m^3/s$, which is not too far from the experimental result, $0.0000156m^3/s$. We now then repeat the same procedure for all the different fluids.

Table 3: Theoretical average flowrate for each different viscosity

Type of fluid	Theoretical average flowrate, $Q_{theoretical}$, m^3/s
Water	0.0000240
30% sucrose solution	0.0000194
50% sucrose solution	0.0000134
55% sucrose solution	0.0000105
60% sucrose solution	0.0000075
65% sucrose solution	0.0000032

5. Graph of viscosity vs flowrate

5.1 Graph of average flowrate vs viscosity

Now, I will plot the graph for the <u>average flowrate vs viscosity</u>. There are no error bars for the theoretical values because they are calculated values and assumed to be correct. Viscosity has an error bar of 5% and the errors for experimental flowrate are taken from the standard deviations *(Table 2)*.

Figure 10: Average flowrate vs viscosity with experimental and theoretical values
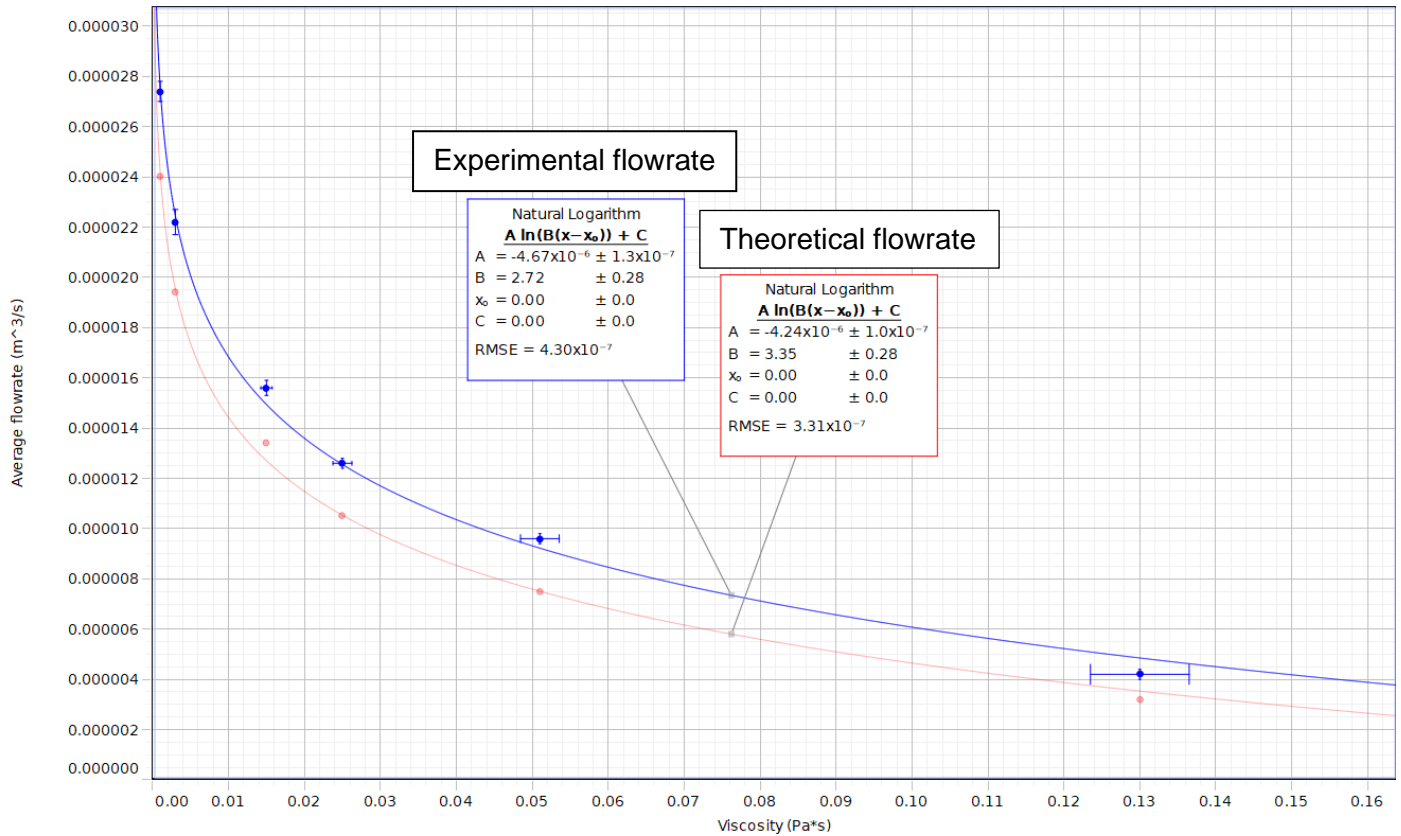
Average flowrate vs viscosity

$$Q_{average\ theoretical}(red) = (-4.24 \times 10^{-6})\ln(3.35\mu) \tag{14}$$

$$Q_{average\ experimental}(blue) = (-4.7 \times 10^{-6} \pm 1 \times 10^{-7})\ln((2.7 \pm 0.3)\mu) \tag{15}$$

<u>Qualitative observations</u>

The equations of *Figure 10* are given above. By using the PASCO Capstone, a logarithmic relationship is found using the software's best-fit function.

Just from looking, the graphs *(Figure 10)* seem to match well. However, are logarithmic relationships really the best fit for these datapoints? According to Hagen-Poiseuille law, the flowrate decreases as viscosity increases, so the decreasing graphs make sense. Yet, there are 3 types of decreasing graphs. Logarithmic

$\ln(x)$, Exponential e^{-x}, Inverse $\frac{1}{x}$. From the *equation 12* that I derived, I ended with a weird proportionality

relation of $Q_1 \propto \frac{1}{t_{final}}(e^{-\frac{t_{final}}{\mu}} - 1)$, in which there are two variables that are changing. According to Hagen-

Poiseuille law (*equation 1*), with $Q_2 \propto \frac{\Delta P}{\mu}$, it also has two variables changing. Hence, from just looking at these

two proportionality equations, it is difficult to tell what kind of equations they will be. However, one thing is

for sure: that the effect of viscosity, μ, on Q_1 and Q_2, could not be pure exponential or inverse as there are two

changing variables. Therefore, although without any mathematical analysis, $\ln(x)$ could be a possible

scenario.

5.2 Further experimental verification of viscosity vs flowrate

It seems from *figure 10* that the theoretical and experimental values of the average flowrate seem to

match. However, is it really the case? Just because the average is the similar, doesn't mean that the trend is. It

doesn't tell the whole story. To verify that, I will plot the sample flowrate vs time graphs for both theoretical

and experimental. In other words, I will show the instantaneous flowrates.

Below is a graph for a trial from 50% solution, and equation could be found using the built-in function from
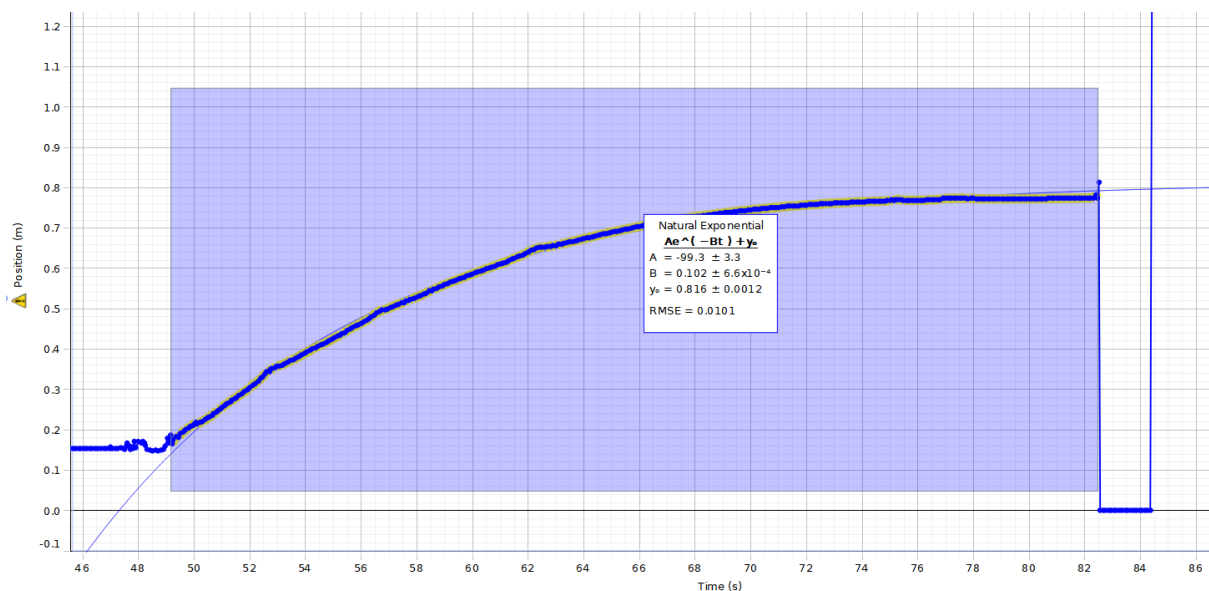
PASCO software.

Figure 11: A trial from the 50% solution

As we can see, for an equation of $y = Ae^{-Bt} + y_0$, B, is -0.102. The A, C and y_0 are irrelevant, because they only explain the position of the graph, but not the rate of change, aka, flowrate of the graph, which is what we are interested in. Hence the experimental B is -0.102. Putting it into our equation of $h(t) = h_0 e^{-Bt}$,

$$h(t) = h_0 e^{-0.102t}, \text{ where } h_0 \text{ is } 0.62m \tag{13}$$

Differentiating with respect to time,

$$\frac{dh}{dt} = |-0.102(0.62)|e^{-0.102t}$$

Adding area (A) to find the experimental flowrate,

$$Flowrate = Q(t)_{experimental} = A\frac{dh}{dt} = \left(\frac{\pi(0.03)^2}{4}\right)0.102(0.62)e^{-0.102t} = 0.0000447e^{-0.102t}$$

Theoretical flowrate:

To find the theoretical flowrate, we use equation (5) from above,

$$Flowrate = Q(t) = A\frac{dh}{dt} = h_0(\frac{\rho g\pi R^4}{8\mu L})e^{-\frac{\rho g\pi R^4}{8\mu LA}t}$$

Plugging in the values,

$$Flowrate = 0.62(\frac{(1228)(9.81)\pi(0.002)^4}{8(0.015)(0.06)})e^{-\frac{(1228)(9.81)\pi(0.002)^4}{8(0.015)(0.06)\frac{\pi(0.03)^2}{4}}t}$$

Giving us the equation for theoretical Q(t),

$$Q(t)_{theoretical} = 0.0000521e^{-0.119t}$$

We then repeat the same process with all the solutions.

5.3 Graphs of instantaneous flowrate

Now that there are equations for both $Q(t)_{theoretical}$ and $Q(t)_{experimental}$, they were graphed using google sheets.

First, both equations were put down and the values at every second were generated. Then, with an appropriate timeframe (40s in this example), the x-y values were plotted separately and put on the same graph. Please refer to *appendix part 1* for all sample calculations.
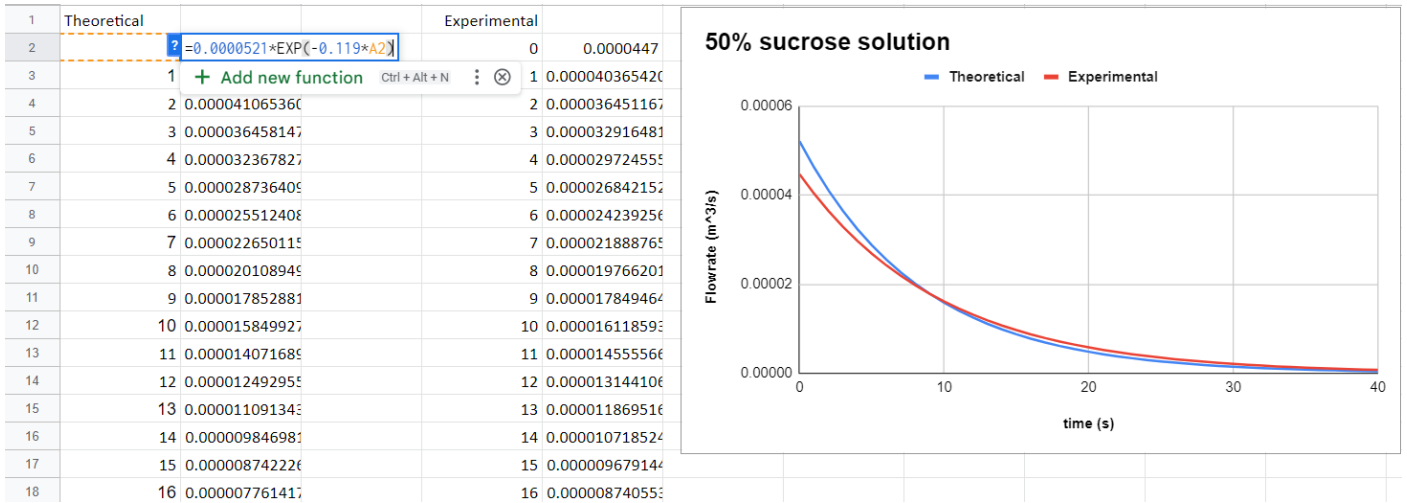
	Theoretical		Experimental	
1				
2	? =0.0000521*EXP(-0.119*A2)		0	0.0000447
3	1 + Add new function Ctrl+Alt+N		1	0.000040365420
4	2 0.000041065360		2	0.000036451167
5	3 0.000036458147		3	0.000032916481
6	4 0.000032367827		4	0.000029724555
7	5 0.000028736409		5	0.000026842152
8	6 0.000025512408		6	0.000024239256
9	7 0.000022650115		7	0.000021888765
10	8 0.000020108949		8	0.000019766201
11	9 0.000017852881		9	0.000017849464
12	10 0.000015849927		10	0.000016118593
13	11 0.000014071689		11	0.000014555566
14	12 0.000012492955		12	0.000013144106
15	13 0.000011091343		13	0.000011869516
16	14 0.000009846981		14	0.000010718524
17	15 0.000008742226		15	0.000009679144
18	16 0.000007761417		16	0.000008740553

Figure 12: An example how the graph was constructed, including equations, generated points and the graph

The following graphs were produced using the method outlined above.

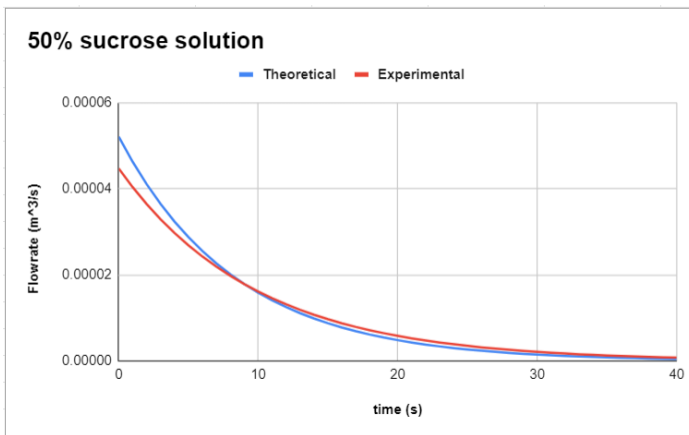*Figure 13: Water solution*   *Figure 14: 30% sucrose solution*
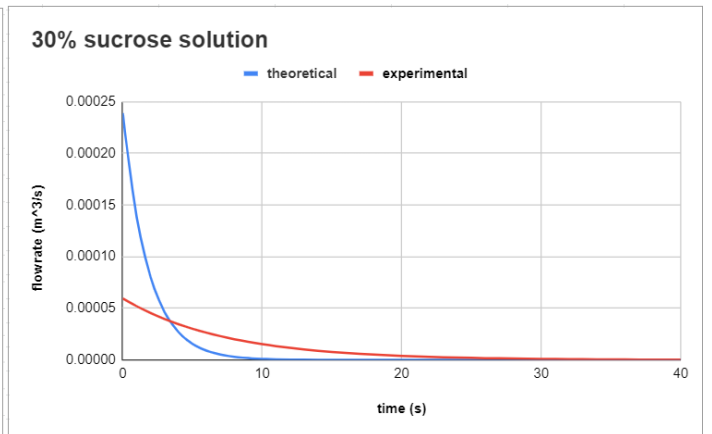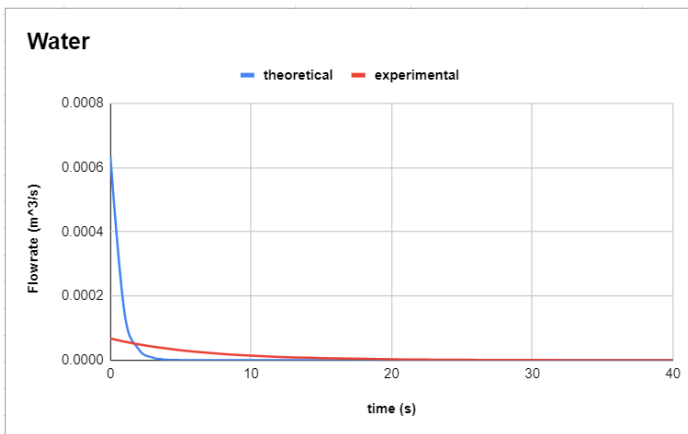
$$Q(t)_{theoretical} = 0.0005635e^{-1.449t}$$

$$Q(t)_{experimental} = 0.0000679e^{-0.155t}$$

$$Q(t)_{theoretical} = 0.0002393e^{-0.546t}$$

$$Q(t)_{experimental} = 0.0000596e^{-0.136t}$$

Figure 15: 50% sucrose solution

Figure 16: 55% sucrose solution

$$Q(t)_{theoretical} = 0.0000521e^{-0.119t}$$
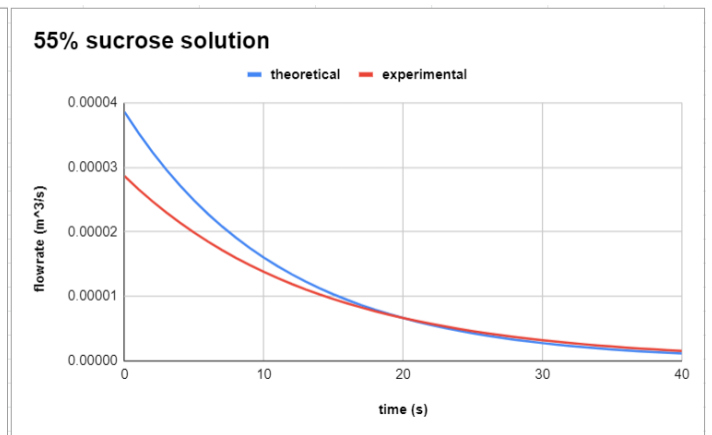
$$Q(t)_{theoretical} = 0.0000321e^{-0.073t}$$

$$Q(t)_{experimental} = 0.0000447e^{-0.102t}$$

$$Q(t)_{experimental} = 0.0000386e^{-0.088t}$$

Figure 17: 60% sucrose solution

Figure 18: 65% sucrose solution
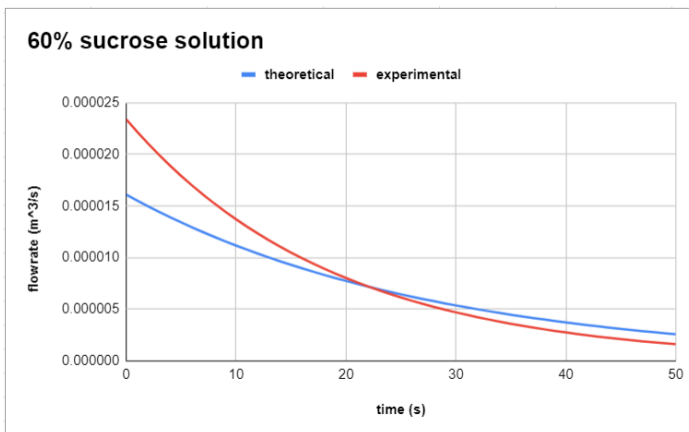
$$Q(t)_{theoretical} = 0.0000161e^{-0.036t}$$
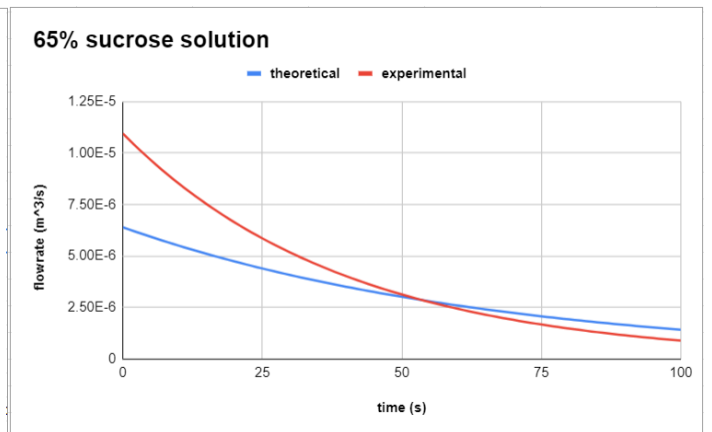
$$Q(t)_{theoretical} = 0.0000064e^{-0.015t}$$

$$Q(t)_{experimental} = 0.0000234e^{-0.054t}$$

$$Q(t)_{experimental} = 0.0000109e^{-0.025t}$$

These graphs are interesting. For the 50% to 65% solutions, the theoretical and experimental graphs seem to match fine: although the bigger errors raised for the 60% and 65%. However, for the water and the 30% one, it seems that the graphs do not match at all. Perhaps, it was only a mere mathematical coincidence that the theoretical and experimental values managed to match?

5.4 Error calculations:

Now, going back to *Figure 10*, errors will be calculated by plotting the theoretical values against the experimental ones. If the data matches perfectly, then the resulting slope should be 1. If not, the appropriate error calculations will be done with the actual slope and the theoretical slope (1).

Figure 19: Error propagation graph
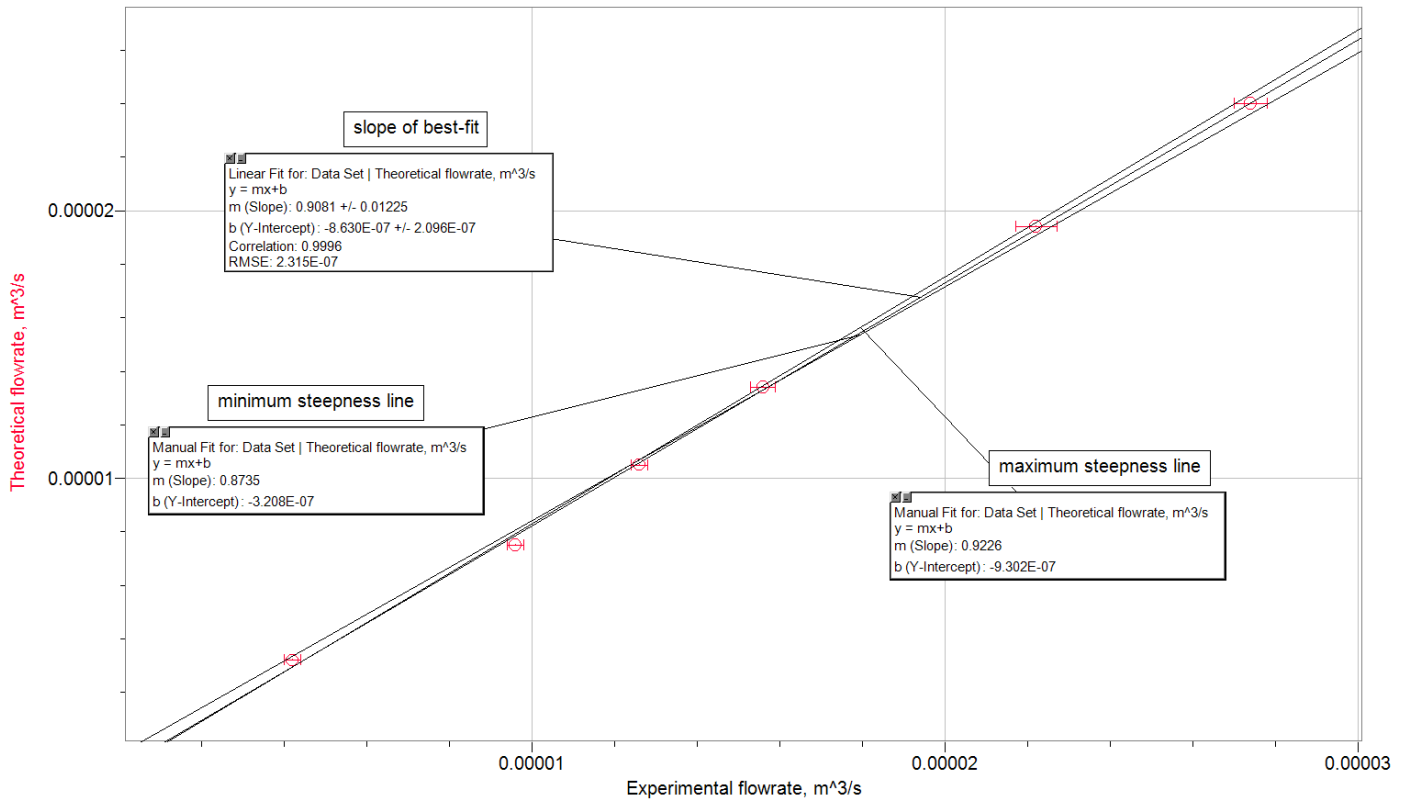
The following slope values are found:

$$m_{best\ fit} = 0.908$$

$$m_{max\ steepness} = 0.923$$

$$m_{min\ steepness} = 0.874$$

Note it is unitless because the axis both have m^3/s, so they cancel out.

Error propagation:

$$m_{best\ fit} \pm \Delta m_{best\ fit} = m_{best\ fit} \pm \frac{m_{max} - m_{min}}{2}$$

$$m_{best\ fit} \pm \Delta m_{best\ fit} = \left(0.908 \pm \frac{0.923 - 0.874}{2}\right)$$

$$m_{best\ fit} \pm \Delta m_{best\ fit} = (0.908 \pm 0.0245)$$

$$m_{best\ fit} \pm \Delta m_{best\ fit} \approx (0.91 \pm 0.03)$$

$$Relative\ error = \frac{absolute\ error}{experiemntal\ value} \times 100\% = \frac{0.03}{0.91} \times 100\% = 3.29\%$$

$$Percent\ error = \frac{|Theoretical\ value - Experimental\ value|}{Theoretical\ value} \times 100\% = \frac{|1 - 0.91|}{1} = 9\%$$

$$Systematic\ error = |Percent\ error - Relative\ error| = |9 - 3.29| = 5.71\%$$

The detailed error propagation steps and results are listed above.

6. Evaluation:

6.1 Evaluation of the experimental set-up

All the equipment used are standard laboratory set-up, such as the lab stand, the tube, the PASCO motion sensor, and an electronic balance. Therefore, the equipment used is accurate. However, the PASCO motion sensor could only take up to 20Hz; when going beyond that speed, the sensor does not register well. This is perhaps because of the rather narrow tube diameter and the transparent liquid. I could have added some black ink (it has minimal effect on viscosity) and see if this will make the motion sensor perform better. In addition, I could use a wider diameter tube.

6.2 Evaluation of the experimental procedure

The first procedure is to make the sucrose solution at different viscosities. The viscosity values were directly taken from other papers, as even though I tried to set up a separate experiment to measure the values of viscosity by myself, they did not lead to an accurate value. I then must trust the external sources (*ISCOTABLES, 7th ed*) for these values. However, I did follow their procedure of making the solution weight by weight, and I use a balance with an uncertainty of ± 0.1g, which is accurate. Another factor was the temperature, but I did use the Arrhenius model to adjust the values accordingly. If I have the equipment, I could use a viscometer.

The wobbling tube is also a source of error. Especially towards the end of the flow, when the flowrate is slow, even a very small wobble could affect the distance reading from the motion sensor.

6.3 Evaluation of the results

The results obtained from this experiment displayed graphically and overall, they are accurate. The average flowrate, as shown in *Figure 10,* matches well with theoretical and experimental values, but when we did further verification with the instantaneous flowrate (*Figure 13-18*), it seems to be simply a mathematical coincidence for water and 30% solution. However, the 50% to 65% solutions are accurate. By doing error propagation on the theoretical and experimental values, there is a relative error of 3.29%, a percent error of 9%, and a systematic error of 5.71%.

6.4 Sources of errors:

Turbulent flow

The first error that arises is the accuracy of the Hagen-Poiseuille equation. As mentioned earlier, the equation requires the flow to be laminar. However, this is not always the case for this experiment. Let's begin with the water trial:

Verifying the flow using the Reynold's number equation, where D is the diameter of the pipe,

$$R_e = \frac{\rho v_{mean} D}{\mu} \tag{16}$$

The mean velocity of the fluid,

$$v_{mean} = \frac{Q_{average}}{A} = \frac{Q_{average}}{\pi R^2} = \frac{0.0000274}{\pi(0.002^2)} = 2.18 m/s$$

Plugging in back to the equation

$$R_e = \frac{(\frac{997kg}{m^3})(2.18m/s)(0.004m)}{(0.001 Pa \cdot S)} = 8694$$

$R_e = 8694 > 2300$, hence turbulent flow.

Table 4: Reynold's number for each viscosity

Fluid	Reynold's number	Flow
water	8694	Turbulent
30% sucrose solution	2655	Transition
50% sucrose solution	407	Laminar
55% sucrose solution	179	Laminar
60% sucrose solution	68	Laminar
65% sucrose solution	13	Laminar

This is one explanation for the errors between our data and the theory for water and 30% solution. So, if the flows are already non-laminar, there should be no reason for the theory to hold. To improve on this, we could model the turbulent flow with the Darcy–Weisbach friction factor and the Colebrook-White equation, but they are simply beyond the scope of this paper.

Viscous flow:

For the Hagen-Poiseuille Law to work, it requires the flow to be viscous. This makes sense since if we decrease the viscosity of the fluid infinitely, it implies that the velocity of the fluid will be infinitely big, which does not make physical sense. Hence, the equation is only valid for viscous flow, in which the viscosity of the fluid has a certain effect. If we look at the fluids I used, it turns out that the water is actually an example of non-

viscous fluid. This explains the errors that arises with the water trial, why the initial theoretical flowrate was much bigger than the experimental one (Theo: $0.0005635 m/s^3$ Exp: $0.0000679 m/s^3$).

However, unlike the Reynold's number for turbulence, there isn't a formal mathematical formula to determine whether a fluid is viscous or not. I can argue that it is the same scenario for 30% solution as well, since its viscosity was relatively low.

Last few seconds of the flow:

Figure 20: Last few seconds of the flow
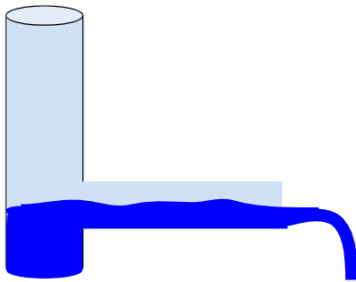
Figure 20 depicts what happens during the last few seconds of the flow. This is when the fluid could not fully fill the tube to produce a parabolic flow. The liquid is still flowing due to pressure and the motion sensor is still on, but it does not obey the Hagen-Poiseuille law. Our model cannot account for these last few seconds, giving systematic errors between the theoretical and experimental values.

Limitation of the theoretical model:

From the theoretical model (*eq.12*) I derived, $Q \propto \dfrac{1}{T_{final}} e^{-\frac{T_{final}}{\mu}}$, I could not isolate the viscosity and write a relationship in terms of one variable. From our reasoning above, the final logarithmic equation seems plausible, but not mathematical deduced.

7. Conclusion:

Throughout the investigation of "Effects of viscosity of a working fluid on the flowrate and the limitations of the Hagen-Poiseuille law", the relationship between viscosity and flowrate is confirmed to be logarithmic using the equation of the best fit from graphical model (*figure 10*).

$$Q_{average\ theoretical} = (-4.24 \times 10^{-6})\ln(3.35\mu)$$

$$Q_{average\ experimental} = (-4.7 \times 10^{-6} \pm 1 \times 10^{-7})\ln((2.7 \pm 0.3)\mu)$$

When calculating the errors between the experimental and the theoretical values, a relative error of 3.29%, a percent error of 9%, and a systematic error of 5.71% are found. Overall, the experimental is accurate when looking at these values.

I set-up an experiment to investigate how the viscosity could affect the flowrate, particularly in a situation where the pressure is not constant. I then verified the results using graphical methods and discussed the limitations of the Hagen-Poiseuille law in case of turbulent and non-viscous flow.

Major sources of errors of the experiment include turbulent flow and non-viscous flow which affect the accuracy of the Hagen-Poiseuille equation. Other errors include the fact that my theoretical model could not account for the last few seconds of the flow, and that it is dependent on two variables: time and viscosity. This paper also relied heavily on external resources for the values of viscosities: although the values were crossed checked with 3 different academic papers, it would have been better if I could use a viscometer to measure the viscosities myself.

I was able to compare my results with another paper (*Drakaki & Sianoudis 2008, European Journal of Physics*), and their results were similar: a decreasing relationship. This paper used more datapoints than mine, which is something that I could improve on.

Areas of further research:

- CFD (Computational Fluid Dynamics) simulations can be used to study and predict the behaviour of fluids at a higher viscosity because it will be unphysical to do so with a motion sensor, as the wait time is too long.

- Studies on how viscosity affects the performance of a hydroelectric generator: Its performance depends on the fluid's flowrate, and viscosity slows it down. Will its performance decrease as well?

- Repeat the experiment with a slow-motion camera and see how the results compare.

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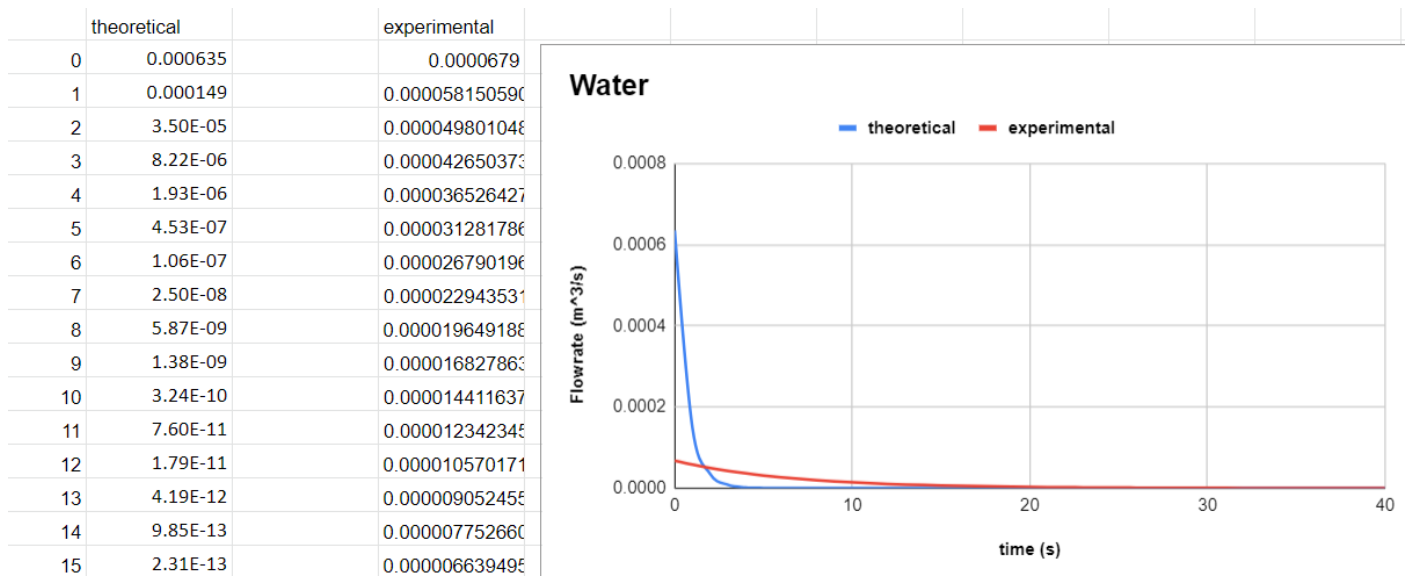
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9. Appendix:

1. **(Part 1)** Detailed images on how *Figure 13-18* were generated using the method outlined in *Figure 12*

Water solution

	theoretical		experimental
0	0.000635		0.0000679
1	0.000149		0.00005815059(
2	3.50E-05		0.00004980104&
3	8.22E-06		0.00004265037:
4	1.93E-06		0.00003652642;
5	4.53E-07		0.00003128178(
6	1.06E-07		0.00002679019(
7	2.50E-08		0.0000229435:
8	5.87E-09		0.00001964918&
9	1.38E-09		0.0000168278 6:
10	3.24E-10		0.000014411637
11	7.60E-11		0.0000123423 4!
12	1.79E-11		0.000010570171
13	4.19E-12		0.00000905245!
14	9.85E-13		0.0000077526 6(
15	2.31E-13		0.0000066394 9!

30% sucrose solution

	theoretical	experimental
0	0.000239	0.0000596
1	0.000138443673	0.00005202142(
2	0.000080195191	0.00004540651:
3	0.000046454045	0.000039632741
4	0.000026909073	0.00003459314(
5	0.000015587410	0.000030194372
6	0.00000902919&	0.00002635493!
7	0.00000523027:	0.000023003711
8	0.00000302969!	0.00002007862(
9	0.00000175499(	0.000017525475
10	0.000001016599	0.000015296982
11	0.00000058887	0.00001335185&
12	0.000000341114	0.000011654071
13	0.00000019759<	0.000010172170
14	0.000000114459	0.00000887870:
15	0.0000000663010	0.000007749711
16	0.00000003840(	0.00000676427&

50% sucrose solution

Theoretical		Experimental	
0	0.0000521	0	0.0000447
1	0.00004625478€	1	0.00004036542(
2	0.00004106536(	2	0.00003645116;
3	0.000036458147	3	0.000032916481
4	0.000032367827	4	0.00002972455!
5	0.00002873640!	5	0.000026842152
6	0.000025512408	6	0.00002423925€
7	0.00002265011!	7	0.00002188876!
8	0.00002010894!	8	0.00001976620:
9	0.000017852881	9	0.00001784946₄
10	0.000015849927	10	0.00001611859;
11	0.00001407168!	11	0.00001455556€
12	0.00001249295!	12	0.00001314410€
13	0.00001109134;	13	0.00001186951€
14	0.00000984698₁	14	0.00001071852₄
15	0.00000874222€	15	0.00000967914₄

55% sucrose solution

	N	O	P	Q	R	S	T	U	V	W	
125											
126		experimental		theoretical							
127	0	0.00003856	0	0.00002861							
128	1	0.0000353117399	1	0.0000265958799							
129	2	0.00003233710!	2	0.00002472355₁							
130	3	0.0000296130599	3	0.00002298303₄							
131	4	0.000027118481	4	0.000021365047							
132	5	0.00002483404₄	5	0.0000198609€€							
133	6	0.0000227420₄€	6	0.0000184627 7(							
134	7	0.00002082627€	7	0.0000171630 0€							
135	8	0.00001907188€	8	0.0000159547₄5							
136	9	0.00001746528!	9	0.000014831544							
137	10	0.0000159940 29	10	0.0000137874 1€							
138	11	0.0000146467 0€	11	0.0000128167 93							
139	12	0.0000134128 8(	12	0.00001191450₁							
140	13	0.0000122829 9₁	13	0.0000110573 0(							
141	14	0.00001124828:	14	0.0000102960 0€							
142	15	0.00001030073;	15	0.00000957117€							
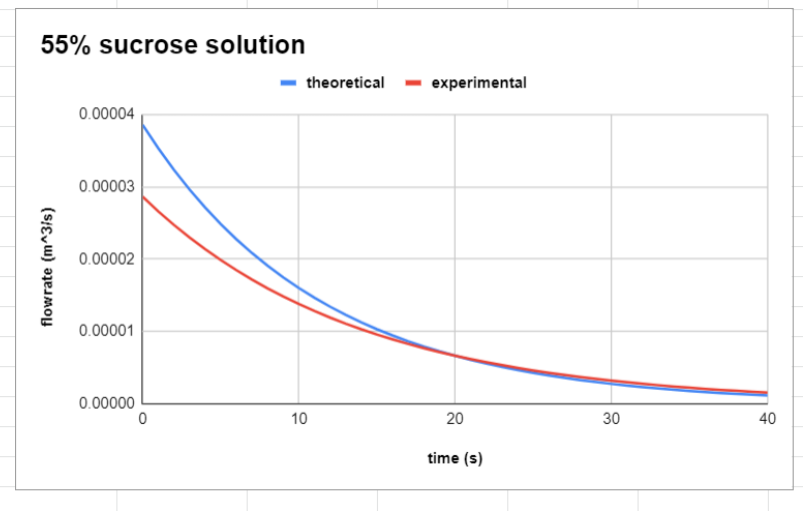
60% sucrose solution

	experimental	theoretical
0	0.0000234	0.0000161
1	0.000022180999	0.00001552139:
2	0.00002102550(	0.0000149635801
3	0.000019930197	0.00001442581^
4	0.000018891952	0.00001390737^
5	0.00001790779:	0.000013407561
6	0.000016974904	0.00001292572^
7	0.000016090612	0.00001246119:
8	0.000015252387	0.0000120133581
9	0.000014457828	0.000011581619
10	0.000013704661	0.000011165395
11	0.00001299072:	0.000010764129
12	0.00001231399(	0.000010377284
13	0.000011672504	0.000010004342
14	0.00001106443(	0.000009644803
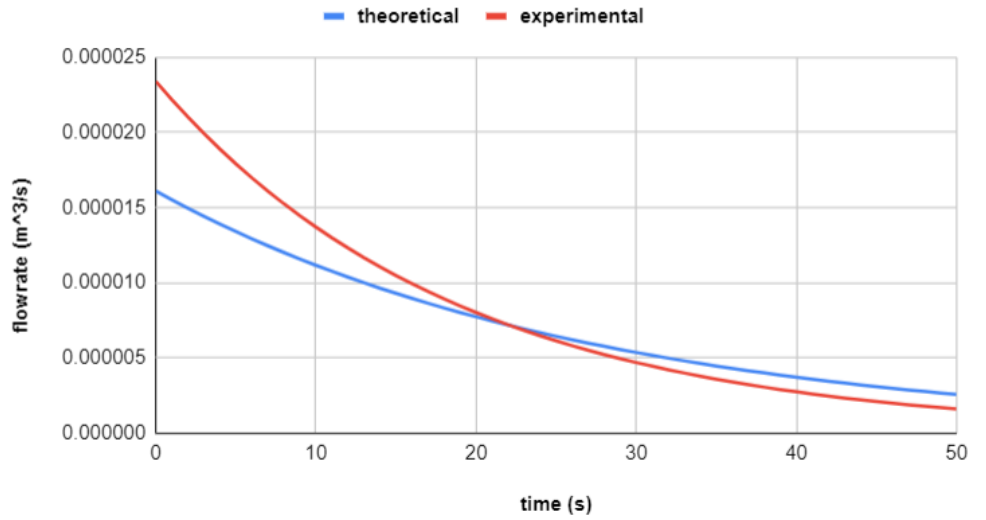
65% sucrose solution

	theoretical	experimental
0	0.0000064	0.00001095
1	0.00000630471(	0.000010679643
2	0.00000621085	0.000010415962
3	0.00000611838	0.000010158791
4	0.00000602729	0.000009907969
5	0.00000593755(	0.000009663341
6	0.00000584915(	0.000009424752
7	0.00000576207(	0.000009192054
8	0.00000567629(	0.000008965101
9	0.00000559178	0.000008743752
10	0.00000550853	0.00000852786(
11	0.00000542651(	0.000008317314
12	0.00000534572(	0.000008111959
13	0.00000526614	0.000007911674