Pin-hole water flow from cylindrical bottles

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We performed an experiment on elementary hydrodynamics. The basic system is a cylindrical bottle from which water flows through a pin-hole located at the bottom of its lateral surface. We measured the speed of the water leaving the pin-hole, as a function of both the time and the current level of water still inside the bottle. The experimental results are compared with the theory. The theoretical treatment is a very simple one based on mass and energy conservation, corresponding to a widespread exercise usually adopted in university basic disciplines of physics.

We extended the previous experiment to another similar system using two identical bottles with equivalent pin-holes. The water flowing from the first bottle feeds the second one located below it. The same concepts of mass and energy conservation now lead to a non-trivial differential equation for the lowest bottle dynamics. We solved this equation both numerically and analytically, comparing the results with the experimental data.

Many university textbooks (see for example [1–7]) refer to the problem of water squirting freely from a small hole in a bottle, as in figure 1. The water speed V at the hole depends on the height H of the free liquid surface above the pin-hole, according to Torricelli's law

$$V^2 = 2gH \tag{1}$$

where g represents the acceleration due to the force of the Earth's gravity. This result was obtained by Galileo's assistant E Torricelli in 1636 [7]. It is normally obtained nowadays through Bernoulli's theorem

$$p + \tfrac{1}{2}\rho V^2 + \rho g y = C \tag{2}$$

introduced by Daniel Bernoulli in his book on hydrodynamics of 1738, which appeared a century

Figure 1. Experimental apparatus used in our test of Torricelli's law. Successive values of the height H were read off from a scale previously glued on to the bottle surface, with points 0.5 cm apart from each other. The corresponding speed of the water flowing at the pin-hole was indirectly measured using the distance X read off the horizontal rule. Successive times T were also measured during the flow.

after Torricelli's law. Here, ρ represents the liquid density, while p, V and y are the local hydrostatic pressure, speed and height, respectively, all measured at the same point inside the liquid. The constant C means that the sum on the left-hand side stays the same, independent of the particular point where the measurements are performed. In our case, one must take two points: the first at the free liquid surface inside the bottle, with $y = H$, neglecting the speed there; and the second point just after the pin-hole, with $y = 0$, outside the bottle. The hydrostatic pressure at these two points is the same, namely the atmospheric pressure. Equation (1) follows directly. This theorem holds for stationary liquid flows, and comes from energy conservation arguments. Thus, in adopting it one is neglecting any energy loss due to viscosity, turbulence, etc.

Because the downward speed of the free liquid surface inside the bottle is neglected, there is a missing factor of $(1 - a^2/A^2)$ on the left-hand side of equation (1), where A and a represent the cross sections of the bottle and the liquid vein (outflow diameter of the pin-hole), respectively. This correction comes from equation (7) to be introduced later on. The cylindrical bottle used in our experiment has a diameter of 9.5 cm, and the pin-hole is made of a thin plastic tube (extracted from a ballpoint pen) whose diameter is 2.0 mm. Thus, the neglected factor of $(1-a^2/A^2)$ represents a relative deviation of the order of 10^{-7}, much smaller than our experimental accuracy. Also, the liquid vein cross section a is a little bit smaller than that of the pin-hole, due to the phenomenon of *vena contracta* [3, 5, 7].

The text is organized as follows. In the next section we describe our first, simplest experiment with a single bottle (figure 1), where we measured simultaneously the speed V and the time T elapsed from the initial height H_0, while the bottle drains off, for successive values of decreasing height H. The speeds are indirectly obtained from the distance X measured on the horizontal rule located at a fixed height below the bottle: X is thus proportional to V. In order to compare our experimental results with the theory, a simple correction is made to Torricelli's law, taking energy losses into account. We also obtain the solution of Bernoulli's differential equation (2) for this case, giving the time evolution of $V(T)$ and $H(T)$ during the flow, comparing them with the experiment. In the subsequent section we describe our second experiment with two identical bottles (figure 2), where the water flowing from the first bottle is caught by the second through a funnel. Initially, the bottles are equally filled, and then both flows start at the same time. Since we had already verified the validity of Torricelli's relation with good accuracy from the first experiment, we used the value determined in this way in the second experiment, instead of measuring the bottom velocity V'. Our accuracy in measuring H (or H') is better than the corresponding accuracy for X (or X' or V'). Thus, in this second experiment we measured only the time evolution of $H'(T)$. In this case, the analytical solution for Bernoulli's differential equation is not trivial, and we solved it numerically. Nevertheless, the analytical solution could be obtained by using some tricks. The results are compared with the experiment. Finally we present our conclusions.

Figure 2. Two-bottle experiment, where the water flowing from the first bottle is caught by the second through a funnel. Successive heights H' were measured as a function of the time.

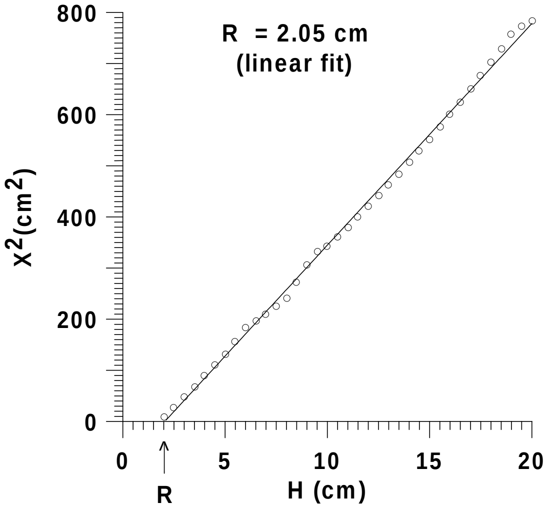

Figure 3. Squared range X^2 versus height H for the one-bottle experiment shown in figure 1. Our estimated error bars are represented by the size of the symbols. The straight line is the best linear fit for the experimental data. The inner diameter of the bottle is 9.5 cm, and that of the pin-hole tube is 2.0 mm. The initial height is H_0 = 20.0 cm, according to which we measured the initial horizontal range X_0 = 28.0 ± 0.3 cm.

The one-bottle experiment

The water flows through a horizontal thin plastic tube installed at the pin-hole. A (parallel) rule is placed below the pin-hole at a distance Y_0. Thus, the 'fall' time of a small 'segment' of water is $\sqrt{2Y_0/g}$. This is the constant of proportionality between the measured distance X in figure 1 and the speed V, i.e.

$$V = \frac{X}{\sqrt{2Y_0/g}}.$$

The water flow is started by one of the authors, responsible for measuring the heights, from an initial value $H_0 = 20.0$ cm, at time $T = 0$. At this same instant, another author starts a chronometer with a memory (actually, a computer program which stores the time elapsed since $T = 0$ each time some key is hit), and a third author reads the value of X off the rule. A fourth author records this value. The whole procedure is repeated at intervals of $H = 0.5$ cm. At the end, one has a three-column table with H, T and X (this last being proportional to the speed V). The initial range $X_0 = 28.0 \pm 0.3$ cm was measured for $Y_0 \simeq 11$ cm (for the sake of clarity, figure 1 shows a larger separation, but our actual measurements were all taken with $Y_0 \simeq 11$ cm), correponding to an initial speed $V_0 \simeq 1.9$ m s^{-1}, supposing $g \simeq 9.8$ m s^{-2}. However, one does not need to perform the transformation from X to V: we will always use X instead of V. In doing so, we do not have to worry about precise measurements for Y_0 and g.

Figure 3 shows the plot of the squared range X^2 versus the height H, which must be a straight line, namely $X^2 = 4Y_0H$, according to Torricelli's law, equation (1). However, our experimental straight line does not cross the origin! On the contrary, the best linear fit for our data, the continuous line $X^2 = K(H - R)$ with optimum values for K and R, crosses the H axis at a minimum residual height $R = 2.05$ cm. Indeed, during the experiment, we noted that the continuous water flow ceases *before* the surface level inside the bottle reaches the pin-hole. This precociously interrupted flow occurs when the height is still around 2 cm. At this point, the surface tension inside the thin tube is enough to compensate the overpressure due to the residual level height, except for isolated drops which start to appear after this point. Even before this, for heights below 5 cm, we could note that the water flow leaving the pin-hole is no longer completely stable, as it was for the initial heights above 5 cm: some oscillations and instabilities appear below that height, as commented on later.

Anyway, equation (1) is not supposed to fit the experimental data, for which we cannot neglect energy losses. Indeed, the sum C in equation (2) is not strictly the same for all positions inside the liquid, but must be smaller at the pin-hole exit than inside the bottle. This behaviour is expected because viscosity and turbulence lead to energy losses when the liquid passes through the thin tube. When applying Bernoulli's equation (2) for a point at the free liquid surface and another point at the pin-hole exit, one must take into account an energy density correction from the former value. We will assume here that this amount is the same throughout the flow, which corresponds to subtracting a constant from the kinetic energy term in equation (2), or, alternatively, subtracting a constant height R from the actual height H. With this correction, Torricelli's law becomes

$$V^2 = 2g(H - R) \tag{3}$$

instead of equation (1), where the residual height R is a constant. Indeed, our experimental data agree quite well with this modified form of Torricelli's law, equation (3), as can be seen in figure 3. Also from figure 3, one can measure the slope $K = 4Y_0 = 43.36$ cm, which is in close agreement with the rule's vertical distance $Y_0 = 10.8$ cm.

The subtracted term $R = 2.05$ cm corresponds to an energy density $\rho g R$ dissipated at the thin tube, i.e. the lost energy per unit volume of liquid. Thus, considering the outflow rate aV, i.e. the volume of liquid crossing the tube per unit time, one obtains an energy dissipation rate, i.e. a dissipated power $\rho g a R V$ proportional to the speed V. This energy dissipation comes from the fact that different points correspond to different liquid speeds: small portions of water near the internal walls, inside the thin tube, move more slowly than those near the tube axis. The relative displacement between adjacent liquid layers dissipates the energy (cylindrical layers inside the thin tube). The constancy of R during

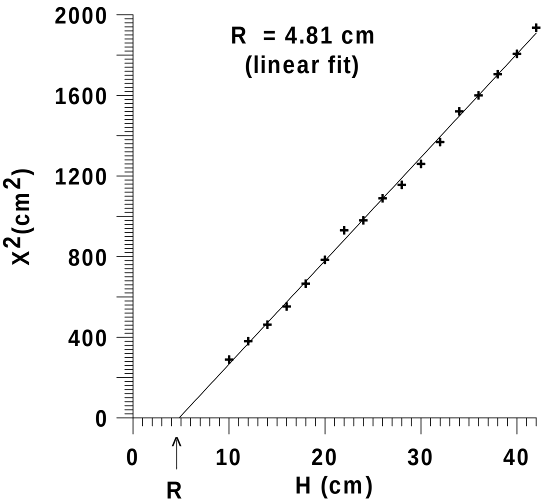

Figure 4. As in figure 3, but for a different bottle with a different pin-hole. The inner diameter of the bottle is 5.1 cm, while that of the pin-hole tube is 3.5 mm. The initial height is H_0 = 42.0 cm, and the corresponding initial horizontal range is $X_0 = 44.0 \pm 0.3$ cm.

the whole flow, experimentally supported by the straight line behaviour in figure 3, means that the energy dissipation occurs according to a constant resultant viscous force $\rho g a R \approx 6 \times 10^{-4}$ N. Of course, the particular value of this force must depend on the liquid's viscosity, and also on the characteristics of the thin tube itself, e.g. its diameter and length, the material of which it is made, etc. Thus, the value of this constant will change depending on the experimental apparatus. In order to test this, we performed the same experiment once again, by using another bottle, now with an inner diameter of 5.1 cm and initial height $H_0 = 42$ cm, with another pin-hole. The experimental results are shown in figure 4. The straight line behaviour agrees again with our modified version of Torricelli's law, equation (3). The thin tube is now metallic, with an inner diameter of 3.5 mm. Accordingly, the new value $R = 4.81$ cm corresponds to a larger resulting viscous force of 5×10^{-3} N, approximately 7 times larger than that of our first bottle. However, this new value is again constant during the whole flow. For this new bottle, we start to observe oscillations and instabilities of the outflow below $H \approx 12$ cm. For $H \approx 8$ cm these instabilities become strong enough to make accurate measurements of X impossible.

It is possible to make our results independent of the particular geometry of the (cylindrical) bottle used during the experiment, provided the condition $a \ll A$ remains valid. In order to do such an universal (bottle-independent) analysis, instead of the quantities H and V denoted by capital letters, we will adopt the following lower case, dimensionless, reduced variables

$$h = \frac{H - R}{H_0 - R} \tag{4}$$

and

$$v = \frac{V}{V_0} = \frac{X}{X_0}. \tag{5}$$

With these definitions, the modified Torricelli's law, equation (3), reads simply

$$v = \sqrt{h}. \tag{6}$$

Figure 5 shows a plot of the experimental values of v versus h, as well as the continuous curve corresponding to the theoretical equation (6). Data from both bottles, presented first separately in figures 3 and 4, are now superimposed into the same plot, defining a single, universal, bottle-independent curve. Except for small deviations at the end of the flow, near the origin in figure 5, the agreement between theory and experiment is quite good. A better test for Torricelli's law is shown in figure 6, where the same experimental data were plotted on a log–log scale. The continuous line is the best linear fit for our first bottle (circles), excluding data taken for $H < 5$ cm, where the quoted flow instabilities and oscillations started to be noted by us during the experiment. The slope of 0.497 we obtained must be compared with the exponent $1/2$ of the square root appearing in equation (6). For the other bottle (crosses), excluding $H < 12$ cm, we obtained a slope of 0.501.

Hereafter, all our remaining experimental results correspond to the bottle geometry shown in figure 1, because the measurements are more accurate in this case (note the number of experimental points in figure 3, more than twice the corresponding number in figure 4). When the stream of liquid crosses the rule height Y_0 below the pin-hole, it is not completely stable, presenting some fluctuations as visible in figure 1. In measuring X, we are forced to estimate an 'average' value at each measurement. Our actual

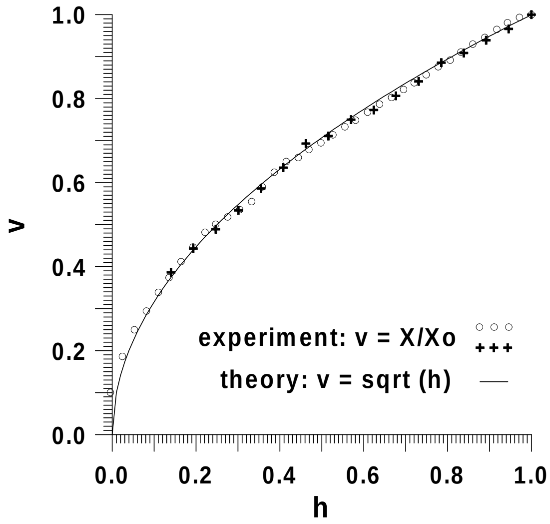

Figure 5. Experimental speed versus height. Different symbols correspond to the same experiment performed with different bottles with different pin-holes, according to figures 3 (circles) and 4 (crosses). By adopting the dimensionless, reduced variables v and h, defined by equations (4) and (5), our results become independent of the particular geometrical dimensions of the bottle and the pin-hole: both data sets can be superimposed into the same, universal, bottle-independent curve. The continuous curve corresponds to the theoretical Torricelli's law, equation (6).

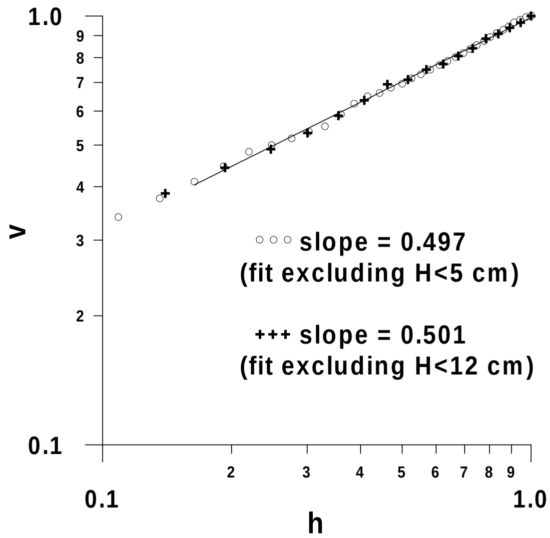

Figure 6. Same experimental data as figure 5. Now, the continuous line is a linear fit for the circles, excluding the last points of the flow where instabilities and oscillations appear. The slopes measured for each different bottle must be compared with the exponent $1/2$ of the square root appearing in equation (6).

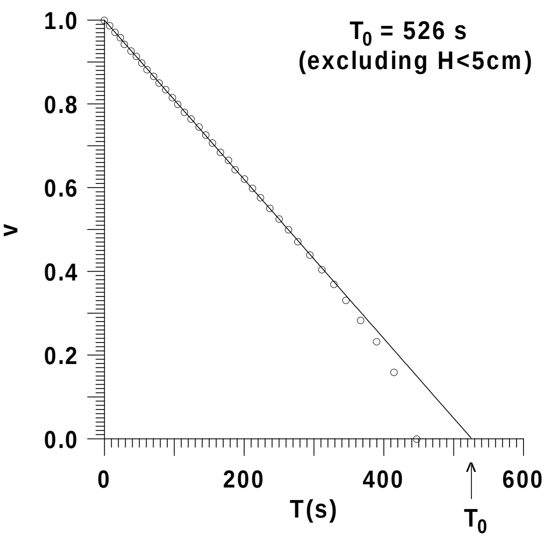

Figure 7. Time evolution of the speed v. The continuous line is the best fit excluding heights $H < 5$ cm, below which an intermittent flow regime starts to be observed.

measurements were made at a vertical height $Y_0 \simeq 11$ cm in order to minimize these fluctuations (in figure 1, the Y_0 distance is exaggerated for clarity). Indeed, the flow seems very stable near the pin-hole, at least for large values of the height H. Since we have already shown good agreement between theory and experiment, concerning the modified Torricelli's law, hereafter we will use equations (4) and (6) in order to obtain the reduced velocity v from the actually measured height H, instead of equation (5) with the measured horizontal range X. The reason for that is the improved accuracy in measuring H at the marks previously glued on the bottle surface, compared with the values of X read off the horizontal rule. From now on, we will study the complete flow dynamics instead of only the instantaneous Torricelli's relation between speed and height.

Figure 7 exhibits the reduced speed v as a function of the time T elapsed since flow started from an initial height $H_0 = 20.0$ cm. The continuous line is the best linear fit, excluding the last six points where instabilities at the pin-hole start to be noted. According to this fit, the complete flow takes a time $T_0 = 526$ s. The real time, however, is smaller than this, something around 450 s, according to the last point in figure 7. For $H < 5$ cm the flow at the pin-hole starts to present small intermittent bursts superimposed on the continuous, stable flow observed previously. This phenomenon is more and more clearly observed as time goes by, until only isolated drops can be observed at the end, below the residual height R, with no traces of any underlying continuous flow. This continuous-plus-intermittent flow is more effective in draining off the bottle faster than would be the previously observed pure-continuous flow, according to the straight line in figure 7. Also, Bernoulli's theorem, even with our constant energy correction subtracted at the pin-hole, is no longer valid: it supposes that one has a *stationary* flow. The theoretical treatment of this intermittency problem is a very difficult task, and many interesting complex phenomena can be observed experimentally [8]. No satisfactory analytical approach is available, although some computer simulations based on a stochastic simple model seem to capture the essential physical ingredients governing the phenomenon [9]—see [10] for a review.

Before the intermittent regime arises below $H \approx 5$ cm, however, Bernoulli's theorem can be applied to our bottle flow, in order to compare the results with the dynamical experimental counterparts shown in figure 7. One also needs to include the conditon that the liquid is not compressible, i.e.

$$aV = -A\frac{\mathrm{d}H}{\mathrm{d}T} \tag{7}$$

where a and A are the cross sections already mentioned. The result is a simple differential equation, namely

$$\frac{\mathrm{d}V}{\mathrm{d}T} = -\frac{a}{A}g. \tag{8}$$

According to this result, the speed V (or v) decreases linearly as time elapses, just as can be seen in the experimental plot, figure 7, before the intermittent regime starts to appear. Also according to this result, the complete draining time would be [5]

$$T_0 = \frac{AV_0}{ag} = \frac{A}{a}\sqrt{\frac{2(H_0 - R)}{g}}. \tag{9}$$

Taking our experimental values $T_0 = 526$ s, $H_0 = 20.0$ cm and $R = 2.05$ cm, adopting $g \simeq$

9.8 m s^{-2}, and considering a bottle diameter of 9.5 cm, we can estimate a. As already mentioned, a is not the cross section of the pin-hole, but that of the liquid vein (*vena contracta*). Indeed, equation (9) gives a diameter of 1.8 mm for the *vena contracta*, which is a little bit smaller than that of the actual pin-hole, namely 2.0 mm.

From now on, we will also use a lower case, dimensionless, reduced variable

$$t = \frac{T}{T_0} \tag{10}$$

when referring to the time. With this notation, the theoretical time evolutions of the reduced speed and height are simply

$$v = 1 - t \tag{11}$$

and

$$h = (1 - t)^2 \tag{12}$$

again independent of the particular (cylindrical) bottle actually used during the experiment. Both the experimental and theoretical values for these evolutions are shown in figure 8. Once more, deviations between theory and experiment are visible at the end, where the flow starts to present an intermittent, non-stationary component. Note, in particular, that the final height ($R \simeq$ 2 cm within the particular geometry of our bottle, corresponding to $h = 0$ in figure 8) is actually reached faster than it would be according to the completely stable flow observed before (theoretical, continuous curve). The intermittency appearing at the end of the whole process helps to drain off the bottle more quickly.

The two-bottle experiment

Considering now the system shown in figure 2, in applying condition (7) to the lowest bottle, one needs to take into account the extra amount of water coming from the upper bottle. It reads now as

$$a(V' - V) = -A\frac{\mathrm{d}H'}{\mathrm{d}T} \tag{13}$$

where the height H' and the speed V' refer to the lowest bottle, being related to each other by the same modified Torricelli's law

$$V'^2 = 2g(H' - R) \tag{14}$$

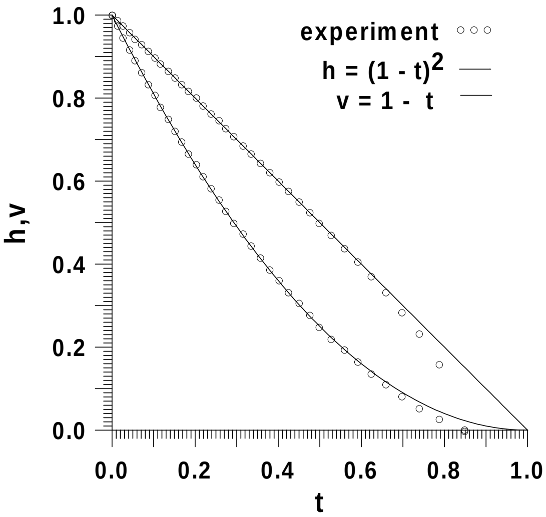

Figure 8. Time evolution of the dimensionless speed v (upper curve) and height h (lower curve), as functions of the dimensionless time, equation (10). Experimental and theoretical values agree very well with each other, except for the last few points where the flow is no longer stationary.

already verified in our first experiment, with $R =$ 2.05 cm. Both the bottles and the pin-hole tubes are identical. We define the reduced speed

$$u = \frac{V'}{V_0} \tag{15}$$

analogously to equation (5). The time evolution of u follows the dimensionless differential equation

$$\left(1 + \frac{\mathrm{d}u}{\mathrm{d}t}\right)u = 1 - t \tag{16}$$

which is the same as equation (13), after being simplified using (14), (15), (9) and (11). Here, t is the same reduced time already defined in equation (10).

One can solve this equation numerically, by dividing the time interval $0 \leqslant t \leqslant 1$ into N equal subintervals. The lower case indices $n = 0, 1, 2, \ldots, N$ represent the discretized times at the borders of these subintervals, i.e. $t_n = n/N$, while u_n represents the value of u at time t_n. The derivative in equation (16) will be replaced by the approximation

$$\frac{\mathrm{d}u}{\mathrm{d}t} \approx \frac{u_{n+1} - u_n}{1/N} \tag{17}$$

where $1/N$ is the size of each subinterval. This approximation is as good as we need, provided a large value for N is adopted. In other words, the numerical solution we will obtain is exact in the sense that one can always keep the errors below any predefined tolerance, no matter how exacting the user who chooses the degree of tolerance is. Equation (17) stands for the derivative of u taken at the centre of the subinterval, i.e. at time

$$t = \frac{n + 1/2}{N} \tag{18}$$

which is the value replacing t on the right-hand side of equation (16). Accordingly, another approximation, namely

$$u \approx \frac{u_{n+1} + u_n}{2} \tag{19}$$

will replace u outside the parentheses on the left-hand side of (16). Then, by performing these replacements, equation (16) is transformed into a second degree equation for u_{n+1} whose solution is

$$u_{n+1} \approx \frac{1}{2N}\left[\sqrt{(2Nu_n - 1)^2 + 8(N - n) - 4} - 1\right]. \tag{20}$$

Now, one needs only to choose a convenient value for N, program this equation on a computer (or even a pocket calculator) and process it starting from $u_0 = 1$ with $n = 0$ on the right-hand side. The result will be the next value, i.e. u_1. Then, repeating the same program with this recently obtained value for u_1 and $n = 1$ on the right-hand side, one gets u_2. After repeating this process N times, one has the complete function $u(t)$ along the interval $0 \leqslant t \leqslant 1$.

The lower bottle is fed by the upper one, up to $t = 1$ when the upper flow ceases. However, at this time the lower bottle is still draining with a reduced speed u^*. From this moment on, there is only one flowing bottle, as in our first experiment. Thus, u will decrease linearly, starting with $u = u^*$ at $t = 1$, at the same constant rate observed in our first experiment, namely $\mathrm{d}u/\mathrm{d}t = -1$. The flow from the lower bottle finally also ends at $t = 1 + u^*$. The value u^* is then a key parameter for our system. By running our program for equation (20) with $N = 100$, we obtained $u^* = 0.546\,301\,91$. Running it again with $N = 1000$, we got $u^* = 0.546\,293\,10$, which means that $N = 100$ is sufficient to achieve accuracy up to the fourth decimal figure, corresponding to a relative deviation of the order of 10^{-5}, far below our experimental accuracy. By running the same program with $N = 10\,000$, and once again with $N = 100\,000$, which takes much less than one second on a PC, we obtained

$$u^* = 0.546\,293\,02 \tag{21}$$

in both cases, meaning that this value is the exact figure, at least up to the eighth decimal figure.

From equation (16) one can verify two limiting cases. First, near $t = 0$, one must have $u \simeq 1 - t^2/2$. Note that the first derivative of u vanishes at $t = 0$, as it must at the beginning because the outflow of the lower bottle is exactly compensated by the inflow coming from the upper bottle (remember that both bottles start from the same initial heights $H_0 = H_0'$). Second, in reaching $t = 1$ from below, one has $\mathrm{d}u/\mathrm{d}t = -1$, compatible with the one-bottle constant flow regime which will be installed from this moment on. Both limiting cases can also be observed in equation (20), namely $u_1 \simeq 1 - 1/2N^2$ and $u_N - u_{N-1} \simeq -1/N$.

Alternatively, one can solve equation (16) analytically, by replacing u by $\omega(1 - t)$, which allows one to separate the variables t and ω into different sides of the resulting equation, i.e.

$$\frac{\mathrm{d}t}{1 - t} = \frac{\omega\,\mathrm{d}\omega}{1 - \omega + \omega^2}. \tag{22}$$

Performing the integration, one finally reaches the ugly analytical solution

$$1 - t = \frac{\exp\left(-\frac{\pi}{6\sqrt{3}}\right)\exp\left[-\frac{1}{\sqrt{3}}tg^{-1}\left(\frac{2\omega-1}{\sqrt{3}}\right)\right]}{\sqrt{\omega^2 - \omega + 1}}. \tag{23}$$

Going back to $u = \omega(1 - t)$, we have plotted u versus t from this solution, a task which requires much more computer time than the whole processing of the numerical solution (20). The resulting plot is indistinguishable from the one obtained numerically with $N = 100$. Also, by taking the limit $t \to 1$ in this solution, we obtained

$$u^* = \exp\left(-\frac{\pi}{3\sqrt{3}}\right) \tag{24}$$

in complete agreement with the numerically obtained value (21).

The above paragraphs of this section concern what the theory says about our two-bottle experiment. We have also measured the successive

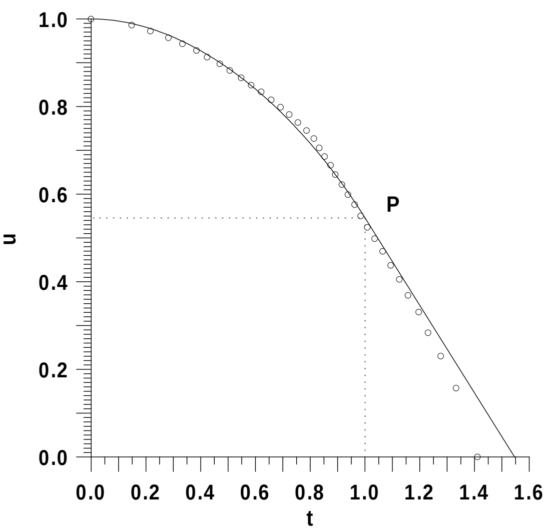

Figure 9. Experimental time evolution of the dimensionless speed u of the lower bottle, as a function of the dimensionless time, in the two-bottles experiment (figure 2). The continuous curve corresponds to the theoretical result.

heights H', from which we obtained the reduced speed

$$u = \sqrt{\frac{H' - R}{H_0 - R}} \tag{25}$$

as a function of time. The result is plotted in figure 9, together with the theoretical continuous curve. The point denoted by P corresponds to u^* at $t = 1$. Our experimental value is

$$u^* = 0.535 \tag{26}$$

obtained by interpolating the two experimental points which are nearest to $t = 1$. The relative deviation between theory and experiment, concerning this value, is only 2%. In performing the experiment, we have been careful to interrupt the upper flow as soon as the top bottle reaches the residual level $R \simeq 2$ cm. After this, only isolated drops would feed the lower bottle, but they were not included in our one-bottle, continuous flow analysis which ends at the residual level R. Thus, this further flow from isolated drops cannot be included in our two-bottle experiment.

Note again deviations at the end of the flow, similar to those that have already appeared in the one-bottle experiment: the last points are located below the theoretical straight line, because the intermittent flow regime accelerates draining. Note also the smaller deviations appearing just before $t = 1$; now, the experimental points lie *above* the theoretical curve. This is so because the accelerated draining of the top bottle, due to the intermittent flow occurring just before $t = 1$, feeds water into the lower bottle *faster* than it would according to the theoretical, continuous flow.

Conclusions

We performed a simple experiment on elementary hydrodynamics, verifying the validity of Torricelli's law, equation (1), relating the height H of the free surface of water in a bottle with the speed V at which the liquid flows through a pinhole. The speeds are indirectly measured through the horizontal range X (figure 1). In order to take energy losses into account, a simple correction was introduced into this law: equation (3) replaces the original form (1). According to our experimental observations (figures 3 and 4) the residual height R subtracted from H in order to take energy losses into account is a constant during all the flow. We have also measured the dynamical evolution of this system, i.e. the dependences of both H and V on the time T, comparing the results with the theoretical framework of Bernoulli's theorem. We also performed another experiment, using two identical bottles, the liquid flowing from the first bottle feeding the second one. In this case, we determined a non-trivial differential equation from the same framework. It was solved both numerically and analytically, and the results were compared with the experimental dynamical evolution.

We believe that such simple experiments can be reproduced by students, in order to better understand elementary hydrodynamic concepts. Also, other similar systems—e.g. those quoted as examples, exercises and problems in references [5–7]—could be experimented on along the same lines presented here. In such experiments, the fixed parameters characterizing each bottle are:

- the initial height H_0;
- the initial speed V_0;
- the residual height R, experimentally obtained through a plot of V^2 versus H (figures 3 and 4); and
- the total time T_0 required to drain the bottle, experimentally obtained through a plot of V versus T (figure 7).

On the other hand, the dynamical variables depending on the time T during the flow are: H and V for our one-bottle experiment (figure 1); or H' and V' for our two-bottles experiment (figure 2). Nevertheless, neither the expressions of these dynamical variables as functions of the time nor the mathematical relation between them would depend on the particular geometry of the cylindrical bottles and pin-holes actually used in the experiments. In order to stress this independence, we adopted the reduced, dimensionless, bottle-independent variables h, v, t and u defined by equations (4), (5), (10) and (15). According to these dimensionless variables, the theoretical equations (6), (11), (12), (20) and (23) can be experimentally tested with other bottles, different from those we used here. We also believe that students should be introduced as often as possible to the very good scientific practice of using universal variables, independent of particular implementations or parameters.

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